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Keywords: Realized volatility, multiplicative volatility models, long memory, international volatility forecasting.

JEL classification: C20, G12

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October 7, 2011

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The volatility specification of the Markov-switching Multifractal (MSM) model is proposed as an alternative mechanism for realized volatility (RV). We estimate the RV-MSM model via Generalized Method of Moments and perform forecasting by means of best linear forecasts derived via the Levinson-Durbin algorithm. The out-of-sample performance of the RV-MSM is compared against other popular time series specifications usually employed to model the dynamics of RV as well as other standard volatility models of asset returns. An intra-day data set for five major international stock market indices is used to evaluate the various models out-of-sample. We find that the RV-MSM seems to improve upon forecasts of its baseline MSM counterparts and many other volatility models in terms of mean squared errors (MSE). While the more conventional RV-ARFIMA model comes out as the most successful model (in terms of the number of cases in which it has the best forecasts for all combinations of forecast horizons and criteria), the new RV-MSM model seems often very close in its performance and in a non-negligible number of cases even dominates over the RV-ARFIMA model.

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1 Introduction

Volatility modeling is of particular importance to finance practitioners and academics as it allows them to develop models applicable to, for instance, risk management, derivative pricing and portfolio allocation. The finance literature has many alternative approaches for modeling and forecasting volatility of asset returns (see Andersen et al. (2006) for a recent review). Traditionally, volatility has been treated as a latent variable whose dynamics is governed by a process from the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) or Stochastic Volatility (SV) families. However, recent studies have formalized the concept of realized volatility (RV) as an alternative avenue for modeling financial volatility (Bandorff-Nielsen and Shephard, 2002a,b; Andersen et al., 2001b, 2003, 2005). In a nutshell, the notion of RV proposes that daily volatility be computed by summing up intra-day squared returns. This approach is supported by the theory of quadratic variation which suggests that RV should provide a consistent and highly efficient non-parametric estimator of asset return volatility over a given discrete interval.

In this article we propose a new mechanism to model the RV dynamics: the volatility specification of the so-called Markov-switching Multifractal (MSM) models introduced by Calvet and Fisher (2001). The proposed Realized Volatility MSM (RV-MSM) model is estimated via Generalized Method of Moments (GMM) using an adaptation of the moment conditions proposed for the baseline MSM in Lux (2008). We generate best linear forecasts of the RV-MSM volatility by means of the Levinson-Durbin algorithm. The performance of the RV-MSM model is compared against other specifications usually applied for the dynamics of RV (Bandorff-Nielsen and Shephard, 2002a; Andersen et al., 2003, 2006). Moreover, we also investigate the performance of the RV models in comparison to other *standard* volatility models such as (FI)GARCH, MSM and SV models.

There are three main reasons usually provided in the literature to motivate RV modeling (Andersen et al., 2006). First, RV measures the realization of volatility by exploiting information in intra-day data without assuming a specific data generating process (Taylor and Xu, 1997). As a result, RV estimates can be used as a natural benchmark when evaluating volatility forecasts and should provide a more accurate assessment of forecasting accuracy than other, noisier benchmarks such as squared returns (Andersen and Bollerslev, 1998; Andersen et al., 2005; Rossi and Gallo, 2006). Second, RV allows analysts to model volatility dynamics with standard time series techniques while still exploiting the intra-day information (Andersen et al., 2003). Third, given its non-parametric nature, RV reduces the 'curse of dimensionality' problem usually encountered in multivariate volatility settings (Bandorff-Nielsen and Shephard, 2004). Thus, with the increasing availability of high frequency data, RV has become an attractive candidate for modeling volatility at the daily frequency due to its promising capabilities for more accurate volatility measurement and forecasting (Andersen et al., 2000, 2001a, 2003; Bandorff-Nielsen and Shephard, 2002a,b).

The MSM models considered here are appropriately adapted versions of the Multifractal Model of Asset Returns (MMAR) due originally to Mandelbrot et al. (1997). The notion of *multifractality* refers to the variations in the *scaling behavior* of various moments or to different degrees of long-term dependence of various moments. Pertinent empirical findings on multifractality have been reported in numerous studies by economists and physicists so that this feature now counts as a well-established stylized fact of financial markets (see for example Cont (2001); Ding et al. (1993); Lux (1996); Mills (1997); Lobato and Savin (1998); Schmitt et al. (1999); Vassilicos et al. (1993)). The MSM models considered in this article account for multifractal volatility via their built-in hierarchical, multiplicative structure with heterogeneous components (Calvet and Fisher, 2004). The forecasting capabilities of MSM models have been studied in a handful of studies by means of Monte Carlo simulations and empirical applications (Calvet and Fisher, 2004; Lux and Kaizoji, 2007; Lux, 2008; Lux and Morales-Arias, 2010a,b). MSM models have shown very promising forecasting performance often leading to forecast gains against other popular models (e.g. GARCH, Fractionally Integrated GARCH (FIGARCH), Markov Switching GARCH, SV) which motivates us to investigate the capabilities of the MSM mechanism for forecasting RV.

A priori, there are two main reasons why RV coupled to the volatility dynamics of the MSM should provide new insights with respect to volatility forecasting. First, the volatility mechanism of MSM models has a built-in intermediate nature between 'true' long-memory and regime-switching processes. Indeed, as demonstrated by Granger and Terasvirta (1999) it is hard to distinguish empirically between 'true' long-memory and regime-switching and even regime-switching models with as few as two different regimes could easily give rise to apparent long memory. MSM models generate what has been called 'long-memory over a finite interval' and in certain limits converge to a process with 'true' long-term dependence. That is, depending on the number of volatility components, a pre-asymptotic hyperbolic decay of the autocorrelation function in the MSM model might be so pronounced as to be practically indistinguishable from 'true' long memory (Liu et al., 2007). Second, the flexible regime-switching nature of the MSM model should also allow to integrate highly volatile periods without resorting to specifically designed regimes (Lux and Kaizoji, 2007).

We use an interesting high frequency data set on 5 major international stock market indices. The latter data set allows us to compute RVs for each market and use them for forecasting future volatility. We evaluate forecasts of the alternative volatility models by means of mean squared errors (MSE) and mean absolute errors (MAE) in three different sub-samples (the turbulent period of mid 2007 to 2009, the rather tranquil period from mid 2005 to mid 2007, and the longer series that combines both of these subsets), using both RV and squared returns as indicators for true volatility. This enables us to analyze to which extent using RV as opposed to squared returns affects forecast errors at alternative volatility regimes. Forecast complementarities between the various models considered are examined via forecast combinations.

To preview some of our results: We find that the RV-MSM seems to improve upon forecasts of its MSM counterparts and other volatility models in terms of mean squared errors (MSE). Counting the number of winning cases across all time horizons and evaluation criteria, the more traditional RV-ARFIMA approach is found to provide the best performance. Nevertheless, RV-LMSM is typically very close to RV-ARFIMA and also has a non-negligible number of cases where it turns out as the winner of the competition of nine different models. We found it particularly valuable in the more turbulent period (2007 to 2009) and for the longer series combining the tranquil pre-2007 period together with the subsequent financial turmoil (2005 to 2009). RV-ARFIMA, in contrast, was an almost uniform winner particularly during the tranquil times before the onset of the financial crisis.

The paper is organized as follows. The next section introduces the basic concepts of volatility modeling and shortly describes the volatility models considered. Section 3 and 4 discuss the data used for our analysis and the forecasting design of our study, respectively. Section 5 presents selected estimation results. Section 6 reports our main findings for the forecast performance of the various single and combined models, and section 7 concludes.

2 Volatility models

In the following section we briefly discuss the general notion of volatility which will provide the background for the subsequent sections on the various dynamic volatility specifications analyzed in this study. Since the MSM volatility specifications are our main concern, we devote most of this section to describing them.

2.1 General notion of volatility

The discussion on the general concept of volatility modeling is based on the exposition by Andersen et al. (2006). Univariate models of volatility usually consider the following specification of financial returns measured over equally spaced discrete points in time $t = 1, ..., T$:

$$y_t = \mu_t + \sigma_t u_t, \tag{1}$$

where $y_t = p_t - p_{t-1}$ with $p_t = \ln P_t$ the logarithmic asset price, $\mu_t = E[y_t|\mathcal{F}_{t-1}]$ and $\sigma_t^2 = \text{Var}[y_t|\mathcal{F}_{t-1}]$ the conditional mean and the conditional variance (volatility), respectively. The information set $\mathcal{F}_{t-1}$ contains all relevant information up to period $t-1$. Moreover, u_t is an independently and identically distributed disturbance with mean zero and variance one. Although u_t can be drawn from various stationary distributions (Chuang et al., 2007), in this study we let $u_t \sim N(0, 1)$. The return components μ_t and σ_t can be specified according to the assumed data generating process. For the purpose of this study we use $\mu_t = \mu + \rho y_{t-1}$. Some conventional specifications for σ_t are, for instance, GARCH and SV. Defining $r_t = y_t - \mu_t$, the latter ‘centered’ returns can be modeled as

$$r_t = \sigma_t u_t. \tag{2}$$

It is important to note that the assumption that all relevant information is contained in $\mathcal{F}_{t-1}$ is strong. In practice, analysts would most likely only be able to observe a *subset* of the information in $\mathcal{F}_{t-1}$. Thus, the ‘true’ volatility σ_t^2 is unobservable, i.e. it is a genuinely latent process.

Andersen and Bollerslev (1998) and Andersen et al. (2005) show that RV obtained from intra-day data is an appropriate estimator for the actual volatility at period t. To concretize the RV notion it is useful to consider a continuous time version of (1):

$$y(t) = \mu(t) + \sigma(t)u(t), \tag{3}$$

where $y(t) = dp(t)$ is the change in the log price, $\mu(t) = q(t)dt$ with $q(t)$ the drift component, $\sigma(t)$ is the ‘spot’ volatility and $u(t) = dw(t)$ with $w(t)$ a standard Brownian motion. Consider now a time interval $[t-1, t]$, then the expression for one-period returns $y(t,1)$ is:

$$y(t,1) = p(t) - p(t-1) = \int_{t-1}^{t} q(s)ds + \int_{t-1}^{t} \sigma(s)dw(s). \tag{4}$$

Assuming that the variation of the drift is an order of magnitude less than the variation of volatility over the interval $[t-1, t]$ (which is consistent with no arbitrage conditions and usually holds empirically), it can be shown that for an arbitrary observation at time t:

$$\mathrm{E}[\sigma_{\mathcal{I}}^2(t)\,|\mathcal{F}_{t-1}] \approx \mathrm{Var}[y(t,1)\,|\mathcal{F}_{t-1}], \tag{5}$$

with

$$\sigma_{\mathcal{I}}^2(t) \equiv \int_{t-1}^{t} \sigma^2(s)ds, \tag{6}$$

the so-called integrated variance (volatility). It follows that the integrated volatility $\sigma_{\mathcal{I}}^2(t)$

provides (theoretically) an accurate ex-post measure of actual volatility in the interval $[t-1, t]$.

Andersen and Bollerslev (1998), Andersen et al. (2001b) and Bandorff-Nielsen and Shephard (2001) show that in practice the empirical counterpart of (6) can be efficiently obtained via RV:

$$\sigma^2_{\mathcal{R}}(t, \Delta) = \sum_{j=1}^{1/\Delta} y^2(t-1+j\cdot\Delta, \Delta), \tag{7}$$

with $0 < \Delta < 1$ and $1/\Delta$ integer. The authors show that RV is a consistent estimator of integrated volatility in the limit, that is:

$$\operatorname*{plim}_{\Delta\to 0} \sigma^2_{\mathcal{R}}(t, \Delta) = \sigma^2_{\mathcal{I}}(t). \tag{8}$$

Andersen et al. (2001b) and Bandorff-Nielsen and Shephard (2001, 2002a) show that returns standardized to the square root of the integrated volatility should (theoretically) follow a Normal distribution. This result was generalized by Andersen et al. (2003) for the multivariate case. Empirical evidence has confirmed these theoretical findings: returns standardized to realized standard deviations are approximately Normally distributed (Andersen et al., 2001b,a).

In what follows we consider the discrete time model in (2) for $t = 1, ..., T$ *daily* time periods. To save on notation, let $\sigma^2_{\mathcal{R},t}$ denote daily RV. A discussion on the computation of RV for this study is given in Section 3. In order to distinguish between RV models and non-RV models (i.e. MSM, SV, (FI)GARCH) we use $\sigma^2_{\mathcal{L},t}$ to denote the latter. Thus, σ_t in (2) is taken from $\sigma_t = \{\sigma_{\mathcal{L},t}, \sigma_{\mathcal{R},t}\}$.

2.2 Markov-switching Multifractal models

We now turn to a description of the MSM model. An in-depth analysis of this model can be found in Calvet and Fisher (2004) and Lux (2008). In the MSM model, instantaneous volatility is determined by the product of k volatility components or multipliers $M_t^{(1)}, M_t^{(2)}, \ldots, M_t^{(k)}$ and a scale factor $\sigma^2_{\mathcal{L}}$:

$$\sigma^2_{\mathcal{L},t} = \sigma^2_{\mathcal{L}} \prod_{i=1}^{k} M_t^{(i)}. \tag{9}$$

Following the basic hierarchical principle of the multifractal approach, each volatility component $M_t^{(i)}$ will be renewed at time t with a probability γ_i depending on its rank within the hierarchy of multipliers, and will remain unchanged with probability $1 - \gamma_i$. Convergence of the discrete-time MSM to a Poisson process in the continuous-time limit requires to formalize transition probabilities according to:

$$\gamma_i = 1 - (1 - \gamma_k)^{(b^{i-k})}, \tag{10}$$

with γ_k and b parameters to be estimated (cf. Calvet and Fisher (2001)). Since we are not interested in the continuous-time limit in this article, we follow Lux (2008) and use pre-specified parameters $\gamma_k = 0.5$ and $b = 2$ in equation (10) and set the number of multipliers $M_t^{(i)}$ to $k = 15$. The choice of parameters for γ_k and b can be motivated by the fact that the in-sample fit and out-of-sample forecasting performance have been found to be almost invariant compared to other (estimated) values (cf. Calvet and Fisher (2004); Lux (2008)). The number of multipliers k can be motivated by previous findings that show that levels beyond $k > 10$ may improve the forecasting capabilities of the MSM for some series and proximity to temporal scaling of empirical data might be closer (Liu et al., 2007; Lux, 2008). Indeed, having 'too many' multipliers is harmless as the other parameter estimates would remain unchanged beyond some threshold and 'superfluous' multipliers would just absorb part of the scale parameters.

The MSM model is a Markov-switching process with 2^k states. The model is fully specified once we have determined the distribution of the volatility components. It is usually assumed that the multipliers $M_t^{(i)}$ follow either a Binomial or a Lognormal distribution. In the MSM framework, only one parameter has to be estimated for the distribution of volatility components, since one would normalize the distribution so that $E[M_t^{(i)}] = 1$.

In the Binomial MSM (BMSM) multipliers $M_t^{(i)}$ are drawn from a Binomial distribution with values m_0 and $2 - m_0$ ($1 \leq m_0 < 2$) with equal probability (Calvet and Fisher, 2004). This configuration guarantees an expectation of unity for all $M_t^{(i)}$. With pre-specified parameters γ_k and b, the BMSM parameters to be estimated boil down to only two, m_0 and $\sigma_{\mathcal{L}}^2$, although the number of states could be arbitrarily large (for large k).

In the Lognormal MSM (LMSM) model, multipliers are determined by random draws from a Lognormal distribution with parameters λ and ν, i.e.

$$M_t^{(i)} \sim LN(-\lambda, \nu^2). \tag{11}$$

Normalization via $E[M_t^{(i)}] = 1$ leads to

$$\exp(-\lambda + 0.5\nu^2) = 1, \tag{12}$$

from which a restriction on the shape parameter ν can be inferred: $\nu = \sqrt{2\lambda}$. Hence, the distribution of volatility components corresponds to a one-parameter family of Lognormals with the normalization restricting the choice of the shape parameter. Thus, the LMSM parameters to be estimated are λ and $\sigma_{\mathcal{L}}^2$.

Lux (2008) introduced a GMM estimator that is universally applicable to all possible specifications of MSM processes. In the GMM framework the unknown parameter vector φ is obtained by minimizing the distance of empirical moments from their theoretical counterparts, i.e.

$$\widehat{\varphi}_T = \arg\min_{\varphi \in \Phi} f_T(\varphi)' A_T f_T(\varphi), \tag{13}$$

with Φ the parameter space, $f_T(\varphi)$ the vector of differences between sample moments and analytical moments, and A_T a positive definite and possibly random weighting matrix. Under standard regularity conditions that are routinely satisfied by the MSM models, the GMM estimator $\widehat{\varphi}_T$ is consistent and asymptotically normal (cf. Harris and Matyas (1999)).[1] The parameter vector is given by $\varphi = (m_0, \sigma_{\mathcal{L}}^2)'$ in the case of the BMSM model and $\varphi = (\lambda, \sigma_{\mathcal{L}}^2)'$ for the LMSM model, respectively.

In order to account for the proximity to long memory characterizing MSM models, Lux (2008) proposed to use logarithmic differences of absolute returns together with the pertinent analytical moment conditions, i.e.

$$\xi_{t,T} = \ln|r_t| - \ln|r_{t-T}|. \tag{14}$$

[1]The standard regularity conditions are problematic for the 'first generation' MMAR model of Mandelbrot et al. (1997) because of its restrictions to a bounded interval. This is not an issue for the 'second generation' MSM of Calvet and Fisher (2001) which is a variant of a Markov-switching model.

Using (2) and (9) in (14) we get the expression

$$\xi_{t,T} = 0.5 \sum_{i=1}^{k} \left(m_t^{(i)} - m_{t-T}^{(i)} \right) + \ln |u_t| - \ln |u_{t-T}| \,, \tag{15}$$

where $m_t^{(i)} = \ln M_t^{(i)}$. The variable $\xi_{t,T}$ only has nonzero autocovariances over a limited number of lags. To exploit the temporal scaling properties of the MSM model, covariances of various orders q over different time horizons are chosen as moment conditions, i.e.

$$\mathrm{Mom}\,(T, q) = E\left[\xi_{t+T,T}^{q} \cdot \xi_{t,T}^{q} \right] , \tag{16}$$

for $q = 1, 2$ and $T = 1, 5, 10, 20$, together with $E\left[r_t^2\right] = \sigma_{\mathcal{L}}^2$ for identification of $\sigma_{\mathcal{L}}^2$.

Out-of-sample forecasting of the MSM model estimated via GMM is performed for the zero-mean time series $Y_{\mathcal{L},t} = r_t^2 - \widehat{\sigma}_{\mathcal{L}}^2$ for l-step ahead horizons, by means of best linear forecasts (cf. Brockwell and Davis (1991), c.5) computed with the generalized Levinson-Durbin algorithm developed by Brockwell and Dahlhaus (2004) (see Lux (2008) for further details).

In order to apply the MSM mechanism for RV, we only consider the volatility specification in (9):

$$\sigma_{\mathcal{R},t}^2 = \sigma_{\mathcal{R}}^2 \prod_{i=1}^{k} M_t^{(i)}. \tag{17}$$

Using $E[M_t^{(i)}] = 1$ results in $E[\sigma_{\mathcal{R},t}^2] = \sigma_{\mathcal{R}}^2$. Therefore, we may obtain an estimate of the scaling factor $\sigma_{\mathcal{R}}^2$ as

$$\widehat{\sigma}_{\mathcal{R}}^2 = T^{-1} \sum_{t=1}^{T} \sigma_{\mathcal{R},t}^2. \tag{18}$$

According to the empirical evidence by Andersen et al. (2001a,b, 2003) realized volatility $\sigma_{\mathcal{R},t}^2$ is well approximated by a Lognormal distribution. To be consistent with this property we restrict ourselves to the LMSM specification to model RV. We call this model a Realized Volatility LMSM model (RV-LMSM henceforth).[2]

[2]Nevertheless, we have also experimented with the Binomial MSM structure to forecast RV. Forecasting results

For the estimation of the Lognormal parameter of the RV-LMSM, we consider in analogy to (14), the logarithmic differences of realized standard deviations, i.e.

$$\zeta_{t,T} = \ln \sigma_{\mathcal{R},t} - \ln \sigma_{\mathcal{R},t-T}. \tag{19}$$

Using (17) we get

$$\zeta_{t,T} = 0.5 \sum_{i=1}^{k} \left(m_t^{(i)} - m_{t-T}^{(i)} \right). \tag{20}$$

Estimation of λ is done via GMM using the pertinent moment conditions for $\zeta_{t,T}$ at various horizons:

$$\text{Mom}\,(T, q) = E\left[\zeta_{t+T,T}^{q} \cdot \zeta_{t,T}^{q}\right], \tag{21}$$

with $q = 1, 2$ and $T = 1, 5, 10, 20$. Based on the estimated model parameters we construct best linear forecasts of the zero mean quantity $Y_{\mathcal{R},t} = \sigma^2_{\mathcal{R},t} - \widehat{\sigma}^2_{\mathcal{R}}$ for l-step ahead horizons (see Appendix A for further details).

2.3 Stochastic Volatility model

In this section we briefly describe the stationary SV model which we also employ in our forecasting analysis. The SV model accounts for autoregressive volatility and a stochastic shock in the volatility process. More precisely:

$$\sigma^2_{\mathcal{L},t} = \exp[h_t], \tag{22}$$

where

$$h_t = \kappa + \psi h_{t-1} + \eta \varepsilon_t \tag{23}$$

are almost identical to the RV-LMSM and can be provided upon request. This is in line with previous findings which show that Lognormal and Binomial specifications of MSM models yield virtually undistinguishable results in most cases (Liu et al., 2007; Lux, 2008).

with $|\psi| < 1$ and $\eta > 0$. The shock in the volatility process ε_t is normally distributed with mean zero and unit variance and is generated independently of the shock u_t in the returns (2).

Several estimation approaches have been proposed for the SV model, such as GMM, Efficient Method of Moments (EMM), Quasi-Maximum Likelihood (QML), Bayesian inference or Markov Chain Monte Carlo (MCMC) methods (Melino and Turnbull, 1990; Harvey et al., 1994; Kim et al., 1997; Gallant et al., 1997; Liesenfeld and Richard, 2003). We consider the relatively simple and robust QML approach proposed by Ruiz (1994). This requires transforming r_t in (2) by taking logarithms of the squares to obtain the linear model:

$$\ln r_t^2 = \mathrm{E}[\ln u_t^2] + h_t + \chi_t, \tag{24}$$

where $\chi_t \equiv \ln u_t^2 - \mathrm{E}[\ln u_t^2]$ is a non-Gaussian, zero mean, white noise disturbance term whose statistical properties depend on the distribution of u_t. In the case that u_t is normally distributed with mean zero and unit variance the mean and variance of $\ln u_t^2$ are $\psi(0.5) - \ln(0.5) \approx -1.27$ and $\pi^2/2$, respectively, where $\psi(\bullet)$ is the Digamma function. Equation (24) coupled with (23) form a linear state space model. Parameters can be estimated by means of QML together with the Kalman filter by treating χ_t as though it were $\mathrm{N}(0, \pi^2/2)$. Estimates $\hat{h}_t$ can be obtained via the Kalman filter. The l-step ahead forecast $\hat{h}_{t+l}$ in the SV model can be obtained recursively from the one-step ahead forecasts $\hat{h}_{t+1}$.

2.4 Generalized Autoregressive Conditional Heteroskedasticity models

The GARCH(1,1) model of Bollerslev (1986) assumes that the volatility dynamics is governed by

$$\sigma_{\mathcal{L},t}^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{\mathcal{L},t-1}^2, \tag{25}$$

where the restrictions on the parameters are $\omega > 0, \alpha, \beta \geq 0$ and $\alpha + \beta < 1$. A well-known stylized fact of financial time series is the so-called leverage effect, which is based on the empirical finding that fluctuations of returns intensify after negative financial news and are less pronounced after positive financial news. To account for this stylized fact we also use the

Threshold GARCH (TGARCH) model of Rabemananjara and Zakoian (1993), i.e.

$$\sigma^2_{\mathcal{L},t} = \omega + \alpha r^2_{t-1} + \alpha^- r^2_{t-1}\mathcal{I}(r_{t-1} < 0) + \beta\sigma^2_{\mathcal{L},t-1}, \tag{26}$$

where $\mathcal{I}(\bullet)$ is an indicator function taking the value of 1 if $(r_{t-1} < 0)$ and 0 otherwise.[3]

The FIGARCH model introduced by Baillie et al. (1996) expands the GARCH variance equation by considering fractional differences. As in the case of (25) we restrict our attention to one lag in both the autoregressive term and in the moving average term. The FIGARCH(1,d,1) model is given by

$$\sigma^2_{\mathcal{L},t} = \omega + \left[1 - \beta L - (1 - \delta L)(1 - L)^d\right] r^2_t + \beta\sigma^2_{\mathcal{L},t-1}, \tag{27}$$

where L is the lag operator, d is the parameter of fractional differentiation and the restrictions on the parameters are $\beta - d \leq \delta \leq (2 - d)/3$ and $d(\delta - 2^{-1}(1 - d)) \leq \beta(d - \beta + \delta)$. In the case of $d = 0$, the FIGARCH model reduces to the standard GARCH(1,1) model. For $0 < d < 1$ the binomial expansion of the fractional difference operator introduces an infinite number of past lags with hyperbolically decaying coefficients. Note that in practice, the infinite number of lags in the FIGARCH model with $0 < d < 1$ must be truncated. We employ a lag truncation of 1000 steps.

Estimation of GARCH, TGARCH and FIGARCH models can be done via QML. The l-period ahead forecasts $\hat{\sigma}^2_{\mathcal{L},t+l}$ for these models can be obtained most easily by recursive substitution of one-step ahead forecasts $\hat{\sigma}^2_{\mathcal{L},t+1}$.

2.5 Earlier Realized Volatility models

The earlier RV literature has considered ARMA(1,1) specifications to model RV dynamics (Andersen et al., 2000; Bandorff-Nielsen and Shephard, 2002a). In this study we use the model proposed by Bandorff-Nielsen and Shephard (2002a) (denoted RV-BNS) which assumes that RV can be decomposed into a latent integrated volatility component and a zero-mean error

[3]Note that introducing the TGARCH model to account for asymmetry of volatility might put the other symmetric families of volatility models at a 'disadvantage'. While elaborating on similar asymmetric variants of those model families would be beyond the scope of this paper, our somewhat 'unfair' comparison allows a certain assessment of the importance of this otherwise neglected fact.

component a_t with variance ς^2:

$$\sigma^2_{\mathcal{R},t} = \sigma^2_{\mathcal{I},t} + \varsigma a_t. \tag{28}$$

Bandorff-Nielsen and Shephard (2002a) show that under the assumption of a constant elasticity of variance (CEV) process or an Ornstein-Uhlenbeck (OU) process for the spot volatility $\sigma^2(t)$, the latent integrated volatility has an ARMA(1,1) representation

$$\sigma^2_{\mathcal{I},t} = \mathcal{V} + \vartheta_t, \tag{29}$$

with

$$\vartheta_t = \psi\vartheta_{t-1} + \phi z_{t-1} + n z_t, \tag{30}$$

where z_t is a mean zero white noise error term with variance n^2 and is generated independently of a_t. The parameters of the RV-BNS can be estimated unbiasedly and efficiently (in a linear sense) by means of Maximum Likelihood together with the Kalman filter. For specific details we refer the reader to Bandorff-Nielsen and Shephard (2002a). Estimates $\hat{\vartheta}_t$ can be obtained via the Kalman filter. The l-step ahead forecast $\hat{\vartheta}_{t+l}$ in the BNS model can be obtained recursively from the one-step ahead forecasts $\hat{\vartheta}_{t+1}$ to obtain $\hat{\sigma}^2_{\mathcal{R},t+l}$.

Subsequent studies have reported empirical evidence on fractional integration of logarithmic RV and suggest modeling the latter by means of fractionally integrated autoregressive models (Andersen et al., 2003, 2005). Along these lines, we employ the so-called Autoregressive Fractionally Integrated Moving Average (ARFIMA) specification to model the dynamics of the logarithmic RV which is given by

$$\Phi(L)(1-L)^d \upsilon_t = \Theta(L)x_t, \tag{31}$$

where $\upsilon_t = \ln \sigma^2_{\mathcal{R},t} - \bar{\upsilon}$ represents the centered logarithmic RV, with $\bar{\upsilon}$ the mean logarithmic RV. Moreover, $\Phi(L)$ and $\Theta(L)$ are the AR and MA polynomials of order p and q respectively, d is the parameter of (fractional) differentiation and x_t is a white noise process. Here we consider both the degenerate case $d = 0$ and the more general case $0 < d < 1$, denoted RV-ARMA$(p, 0, q)$ and RV-ARFIMA(p, d, q), respectively.

The AR(FI)MA models can be estimated via Maximum Likelihood. The orders p and q of the RV-ARMA$(p, 0, q)$ are chosen by estimating the model with different orders and selecting the best specification via the Akaike Information Criterion (AIC). In the case of the RV-ARFIMA model we restrict the (p, q) order to a maximum of $p = 1$ and $q = 1$ to save on computation time, i.e. we considered only the combinations (p, q) given by $(0, 0)$,$(1, 0)$,$(0, 1)$ and $(1, 1)$, and chose the order of the model via AIC. Following Lux and Kaizoji (2007), the RV-ARFIMA is estimated in two steps. The fractional differentiation parameter d is first estimated via the Geweke and Porter-Hudak (GPH) periodogram regression and then the remaining AR and MA parameters are estimated by means of the Fox and Taqqu method assuming lag polynomials with roots strictly greater than 1 in modulus. Best linear forecasts can again be obtained from the analytical covariances of the processes and the generalized Levinson-Durbin algorithm of Brockwell and Dahlhaus (2004).

3 The data

Our dataset comprises 3 European stock market indices and 2 North-American indices, whose characteristics are summarized in Table 1. The French index CAC 40 measures the performance of the 40 largest companies listed in Euronext Paris in terms of order book volume and market capitalization. The DAX and FTSE 100 are the main benchmarks for the German and the British stock markets, respectively. These indices include the most highly capitalized blue chip companies in these countries. The US index NYSE Composite tracks all stocks listed on the New York Stock Exchange. Our second American index is the S&P 500 which includes the 500 stocks with the largest market capitalization actively traded on either of the two largest stock markets in the US, NYSE and NASDAQ. Detailed information on the timespan, trading hours, sampling intervals and number of observations per index can be found in Table 1. The DAX dataset was provided by courtesy of Deutsche Börse AG. The other datasets were purchased from Tickdatamarket (*www.tickdatamarket.com*). Intraday data are typically error-prone (Müller et al., 1990). Therefore we had to 'clean' the data before we were able to use it. To save on space, more details on the cleaning procedure can be found in the Appendix.

We employ a time scale $t \in \mathbb{R}_+$ in days, which incorporates only the official trading hours for each index respectively, also called *business time* (Dacorogna et al., 2001). Daily returns and realized volatilities are computed using 30-minute intervals as proposed by Andersen et al. (2003). Formally, daily returns are given by

$$y_t = \sum_{j=1}^{1/\Delta} y_{t-1+j\cdot\Delta,\Delta}, \tag{32}$$

and realized volatilities by

$$\sigma^2_{\mathcal{R},t} = \sum_{j=1}^{1/\Delta} y^2_{t-1+j\cdot\Delta,\Delta}, \tag{33}$$

with $\Delta = 30$ minutes. Standardized returns are given by

$$\tilde{y}_t = \frac{y_t}{\sqrt{\sigma^2_{\mathcal{R},t}}}.$$

Figures 1 to 4 about here

Some descriptive statistics for these quantities are provided in Table 2.

As is usually the case, returns show considerable deviations from the Normal distribution in terms of high excess kurtosis (see Figures 2 and 3). Some index returns are also left skewed, particularly in the case of the S&P 500.[4] By contrast, standardized returns are symmetrically distributed and exhibit kurtosis close to 3: the kurtosis of a Normal distribution. Distributions of standardized returns are approximately Gaussian as confirmed by the kernel density estimates in Figure 3. As displayed in Figure 2, the standardization of returns to realized standard deviations reduces the deviations from the Normal distribution considerably for all indices. RV estimates are rightly skewed and highly leptokurtic, whereas the distribution of logarithmic RV

[4] The S&P 500 data comprises the stock market crash of 1987 when the index dropped 20.4% on a single day, the so called Black Monday. The NYSE Composite data omits the peak of this crash.

estimates can be well approximated by a Normal distribution (see Table 2 and Figure 4). These empirical findings corroborate previous evidence by Andersen et al. (2000, 2001a,b, 2003) and Bandorff-Nielsen and Shephard (2002a) and also support the hypothesis of a Normal mixture distribution of returns as outlined in Section 2.1.

Tables 1 and 2 about here

4 The forecasting design

We employ a forecasting scheme where we estimate a particular model in sample with data up to time $t = S$ and use the estimated parameter space Γ of the model to obtain forecasts $S + l$ recursively for horizons $l = 1, 20, 50, 100$.[5] We have broken down the analysis to three different sample periods for out-of-sample evaluation. The first sample runs from July 2007 to April 2009 denoted the 'turbulent' sample, which includes the period of the 2008-2009 financial crisis. The second sample runs from July 2005 to July 2007 denoted the 'tranquil' sample which excludes the period of the financial crisis. Lastly, the third sample runs from July 2005 to April 2009 denoted the 'tranquil-turbulent' sample as it includes both the period pre- and post-financial crisis.

In what follows, let $\tau = 1, ..., \mathcal{T}$ be an out-of-sample observation with $\mathcal{T}$ the total number of observations. We evaluate volatility forecasts by means of relative MSEs and MAEs based upon two different estimates of the 'true' volatility, namely, squared returns r_τ^2 and RV $\sigma^2_{\mathcal{R},\tau}$.[6] Squared returns r_τ^2 are a popular proxy of 'true' volatility used in the asset volatility literature to evaluate forecast errors. This stems from the fact that, assuming that we have a full information

[5]We have also experimented with rolling window and recursive schemes for estimation of parameters and subsequent forecasting. However, rolling and recursive estimations of some of the models (FIGARCH and RV-ARFIMA) were very time consuming and the results were qualitatively similar when comparing models against each other. We have also computed so-called Mincer-Zarnowitz regressions of volatility estimates on their forecasts. However, results from this criterion turned out to be very inconclusive. This is certainly at least partially due to the violation of the standard assumption for a regression framework by volatility measurements, and hence we put less weight on these findings.

[6]Note that r_τ^2 are the squared residuals obtained after linear filtering of returns y_τ using the in-sample mean and the first-order autocorrelation (see Section 2.1).

set and correctly specified models for volatility, the squared returns should provide an unbiased estimator of volatility. To see this, let r_τ be an arbitrary out-of-sample observation. Forecast evaluation of volatility can be done by comparing volatility forecasts $\hat{\sigma}_\tau^2$ conditional on $\tau - 1$ to the squared returns r_τ^2 since

$$E\left[r_\tau^2 \,|\mathcal{F}_{\tau-1}\right] = E\left[\sigma_\tau^2 u_\tau^2 \,|\mathcal{F}_{\tau-1}\right] = \sigma_\tau^2. \tag{34}$$

However, in practice, r_τ^2 can be a poor ex-post indicator of actual volatility due to its large noise component. In contrast, as discussed in Section 2.1, RV $\sigma_{\mathcal{R},\tau}^2$ should (theoretically) provide a more accurate ex-post measure of actual volatility at period τ. In a nutshell, this occurs as the finer sampling of intra-day returns will eventually annihilate the measurement error.

In what follows, let '0' indicate a benchmark (historical volatility) and '$\bullet$' a particular competing volatility model (BMSM, LMSM, SV, GARCH, TGARCH, FIGARCH, RV-LMSM, RV-BNS, RV-AR(FI)MA). Forecast errors denoted $\hat{e}_\tau(0)$ and $\hat{e}_\tau(\bullet)$ computed against squared returns are given by

$$\hat{e}_\tau(0) = r_\tau^2 - \hat{\sigma}^2, \;\; \hat{e}_\tau(\bullet) = r_\tau^2 - \hat{\sigma}_\tau^2, \tag{35}$$

where $\hat{\sigma}_\tau^2 = \left\{\hat{\sigma}_{\mathcal{L},\tau}^2, \hat{\sigma}_{\mathcal{R},\tau}^2\right\}$ denotes the volatility forecast of the competing model. Alternatively, we consider RV as an indicator for actual volatility along the lines proposed by Andersen and Bollerslev (1998), i.e.

$$\hat{e}_\tau(0) = \sigma_{\mathcal{R},\tau}^2 - \hat{\sigma}^2, \;\; \hat{e}_\tau(\bullet) = \sigma_{\mathcal{R},\tau}^2 - \hat{\sigma}_\tau^2. \tag{36}$$

The MSE and MAE of the benchmark specification are:

$$\bar{d}(0) = \mathcal{T}^{-1} \sum_{\tau=1}^{\mathcal{T}} d_\tau(0), \tag{37}$$

with $d_\tau(0) = \hat{e}_\tau(0)^2$ for MSE and $d_\tau(0) = |\hat{e}_\tau(0)|$ for MAE. The average performance of a competing model specification is given in relation to $\bar{d}(0)$ yielding relative MSE or MAE:

$$\overline{dr}(\bullet) = \frac{\bar{d}(\bullet)}{\bar{d}(0)}, \tag{38}$$

with $\bar{d}(\bullet)$ denoting the MSE or the MAE of a particular model '$\bullet$' defined as in (37).

Table 3 about here

5 Parameter estimates

A detailed account of our empirical estimates in the various standard volatility and RV models is available upon request. Table 4 exhibits estimated parameters of the relatively recent BMSM and LMSM models as well as in our new proposal of a multifractal model for realized volatility, RV-LMSM. Compared with previous studies (Calvet and Fisher, 2004; Lux, 2008) multifractal parameters m_0 and λ are found to be of similar magnitude. Comparison of LMSM and RV-LMSM also shows quite the same range of the estimated parameters between about 0.02 and 0.07 which indicates that the degree of heterogeneity of volatility fluctuations is about the same in both the squared returns and realized volatility series. Parameter estimates for the more standard models appear perfectly in line with previous literature.

Table 4 about here

6 Forecasting results

In what follows we discuss the results of our forecasting analysis. Tables 5 to 9 display the main out-of-sample results for the 5 different stock market indices considered: CAC 40, DAX, FTSE 100, NYSE Composite and S&P 500. Each table displays the results of the relative MSE or MAE for each of the forecasting horizons $l = 1, 20, 50, 100$ measured against squared returns (upper entry) and realized volatility (lower entry). Results for the three sub-samples analyzed are also presented in each table. In the following discussion we first consider results for single models M1 to M10 and subsequently consider the results for some combined forecasts.

6.1 Single models

Starting with the French index CAC 40 (Table 5), we find quite supportive results for the forecast capabilities of various versions of multifractal models. For the complete sample (2005 to 2009), BMSM and LMSM mostly dominate over the other standard volatility models, both when evaluated by realized volatility and by squared returns. However, for both criteria, the RV-ARFIMA and RV-LMSM models typically provide even better forecasts. While dominance shifts between RV-ARFIMA and RV-LMSM under an MSE evaluation, the ARFIMA variant is more clearly the winner under the MAE evaluation albeit with RV-LMSM a close second.

Looking at the tranquil subperiod, we see that RV-ARFIMA dominates for both the MSE and MAE criterion with both volatility measurements and over all forecast horizons, while the FIGARCH model is a similarly clear second winner. However, a glance through the rows and columns of this panel shows that the performance of all models is pretty close (except perhaps for the RV-BNS) so that the gain from the best performing approach is not too important at least for lower lags.

For the volatile period, results are much less clear-cut. While there is some variation across criteria and time horizons, the baseline MSM models most often come first among the standard volatility models while RV-LMSM comes first for a number of cases among the RV models and mostly also dominates BMSM and LMSM.

Note also that both MSEs and MAEs are way higher throughout for the volatile period compared to the tranquil period indicating lower predictability of asset price fluctuations in the former.

For the German DAX (Table 6) overall patterns are not too different from those in the French index. Some differences are that relative performance in the volatile period is even less systematic than for the previous series, and that stochastic volatility shows a better performance than before.

For the tranquil period, again FIGARCH and RV-ARFIMA are the uniformly best models.

Over the complete sample, RV-LMSM performs mostly best under the MSE criterion, while RV-ARFIMA is somewhat better under the MAE criterion.

In the case of the FTSE 100 (Table 7) both the volatile and tranquil subperiods show very diffuse results across models and time horizons. For the complete sample, however, the emerging patterns are close to those of the previous series. One particular feature is that BMSM and LMSM perform mostly better than the other time series models and often even dominate models based on RV.

Broadly similar results are obtained for the NYSE Composite (Table 8) and the S&P 500 (Table 9). Here TGARCH (model M3) appears as the best time series model in the volatile period, while stochastic volatility dominates mostly for the tranquil period. In the volatile period, there is no clear dominance of RV-based models over standard models, while the tranquil period sees RV-ARFIMA followed closely by RV-LMSM dominating all standard models. For the complete sample, RV-LMSM dominates over RV-ARFIMA under the MSE criterion and vice versa for the MAE criterion. Note, however, that in contrast to the other markets, the advantage of RV-based forecasts over standard models is less pronounced.

Tables 5 to 9 about here

6.2 Combined forecasts

A particular insight from the methodological literature on forecasting is that it is often preferable to combine alternative forecasts in a linear fashion (cf. Granger (1989), Aiolfi and Timmermann (2006)). In fact, a recent study by Patton and Sheppard (2009) shows that combining realized volatility estimators could improve forecasting accuracy. In this study we address the issue of forecast complementarities via combined forecasts. The forecast combinations are computed by simple averaging which have been found to work well in relation to other more sophisticated weighting methods (Newbold and Harvey, 2002).

The first forecast combination strategy considered is that of GARCH, TGARCH, LMSM and SV models (C1). This combination strategy might allow us to analyze the complementarities that arise when combining forecasts of models that account for leverage effects, autoregressive volatility, regime-switching, multifractality and 'apparent' long memory.

Forecast combination C2 considers complementarities of the models used for modeling RV dynamics. The combination strategy C2 (RV-ARMA, RV-ARFIMA, RV-LMSM, RV-BNS) could provide an indication of the complementarities that arise when combining models that account for features such as autoregressive volatility, regime switching, multifractality and genuine long memory.

The last set of forecast combinations considered is C3 which is designed to exploit complementarities between standard and realized volatility models. More precisely, the combination strategy C3 (GARCH, BMSM, RV-LMSM, RV-BNS) considers complementarities between RV and non-RV models that account for various features: long memory, autoregressive volatility, regime-switching and multifractality.

As in the case of single models, we generally find lower MSEs and MAEs at short to medium horizons, when they are computed with realized volatility as benchmark as opposed to squared returns. Overall, however, we see very little improvement compared to single models, be it standard time series models or RV-based models. Apparently, there is little gain in averaging the forecasts from different volatility models for our samples. If anything can be said, it is that the combination of RV-based models (C2) often provides better forecasts than combinations C1 and C3. Hence it seems that averaging the standard and RV-based models provides hardly any gain, while RV models (single models or their averages) perform better than standard models in most cases.

7 Conclusion

RV has been recently introduced in the volatility literature as an alternative way to measure and model volatility. The RV-AR(FI)MA and RV-BNS models are some of the most popular models used in the RV literature to model the dynamics of RV and to forecast future (realized) volatility. In this article we propose the RV-LMSM model as a new alternative for modeling and forecasting RV. The RV-LMSM model accounts for two important stylized facts of asset markets which the

RV-BNS and RV-AR(FI)MA ignore: regime-switching and multifractality. We employ the RV-LMSM model to forecast RV for 2 North American stock indices and 3 European stock indices. We compare the forecasting capabilities of the RV-LMSM model against standard volatility models. We also analyze complementarities between all models considered by means of forecast combinations.

In most cases, we find that the RV-LMSM model performs better than non-RV models ((FI/T)GARCH, SV and MSM) in terms of MSEs for most stock indices and at most forecasting horizons in the turbulent or the tranquil-turbulent periods. While RV-ARFIMA has the largest number of cases in which it dominates, there is a certain number of instances in which the RV-LMSM model seems to outperform the popular RV-ARFIMA model in terms of MSEs at various forecasting horizons and for various stock markets in the turbulent and particularly over the complete sample mixing turbulent and tranquil subperiods.

Our results also confirm that using RV as opposed to squared returns as indicators of true volatility when evaluating forecasts errors usually leads to lower MSEs and MAEs particularly at lower to medium horizons. This result sheds light on the importance of using RV estimates when evaluating the forecasting capabilities of models as proposed in previous studies (Andersen et al., 2005; Rossi and Gallo, 2006). Somewhat in contrast to other recent findings we found that forecast combinations of alternative models (non-RV and RV) could hardly improve upon forecasts of various single models. An interesting extension to this study would be to consider multivariate volatility models (non-RV and RV) and apply them to our dataset for performance comparisons. It would certainly also be worthwhile to explore other methods of forecast combinations than simple model averaging. We leave these issues for future research.

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A Moments of the RV-LMSM model for GMM estimation and best linear forecasting

In this section we summarize the closed-form solutions for selected moments for the estimation of the LMSM model via GMM as provided in Lux (2008). Specifics about the derivation of these moments not detailed here may be looked up in the latter paper. We employ these moments in the derivation of the moment conditions for the RV-LMSM.

Consider the product of volatility components

$$\theta_t = \prod_{i=1}^{k} M_t^{(i)}, \tag{39}$$

and its logarithmic increments:

$$\varpi_{t,T} = \ln(\theta_t) - \ln(\theta_{t-T}) = \sum_{i=1}^{k} \left(m_t^{(i)} - m_{t-T}^{(i)} \right), \tag{40}$$

where $m_t^{(i)} = \ln M_t^{(i)}$. The LMSM moments are:

$$E\left[\varpi_{t+T,T}\varpi_{t,T}\right] = -\sum_{i=1}^{k} \left(1-(1-\gamma_i)^T\right)^2 \nu^2, \tag{41}$$

$$\begin{aligned} E\left[\varpi_{t+T,T}^2\varpi_{t,T}^2\right] &= \sum_{i=1}^{k} \left(1-(1-\gamma_i)^T\right)^2 \cdot 6\nu^4 \\ &+ \left\{ \sum_{i=1}^{k} \left(\left(1-(1-\gamma_i)^T\right) \sum_{j=1,j\neq i}^{k} \left(1-(1-\gamma_j)^T\right) \right) \right\} \cdot 4\nu^4 \\ &+ \left\{ \sum_{i=1}^{k} \left(\left(1-(1-\gamma_i)^T\right)^2 \sum_{j=1,j\neq i}^{k} \left(1-(1-\gamma_j)^T\right)^2 \right) \right\} 2\nu^4, \end{aligned} \tag{42}$$

and

$$E\left[\varpi_{t+T,T}^2\right] = \sum_{j=1}^{k} \left(1-(1-\gamma_i)^T\right) \cdot 2\nu^2, \tag{43}$$

with ν the estimated parameter of the normalized Lognormal distribution (cf. eq. 12).

Now recalling:

$$\zeta_{t,T} = \ln \sigma_{\mathcal{R},t} - \ln \sigma_{\mathcal{R},t-T} = 0.5 \sum_{i=1}^{k} \left(m_t^{(i)} - m_{t-T}^{(i)} \right),$$

we have:

$$\zeta_{t,T} = 0.5 \varpi_{t,T}.$$

It follows that

$$E\left[\zeta_{t+T,T}\zeta_{t,T}\right] = 0.25 \cdot E\left[\varpi_{t+T,T}\varpi_{t,T}\right], \tag{44}$$

and

$$E\left[\zeta_{t+T,T}^2\zeta_{t,T}^2\right] = 0.25^2 \cdot E\left[\varpi_{t+T,T}^2\varpi_{t,T}^2\right]. \tag{45}$$

Inserting (41)-(43) in (44) and (45), we obtain the analytical expressions for the moment conditions in (21) employed in the GMM estimation of the RV-LMSM model.

Best linear l-step forecasts of the quantity $Y_{\mathcal{R},t} = \sigma_{\mathcal{R},t}^2 - \widehat{\sigma}_{\mathcal{R}}^2$ are constructed as

$$\widehat{Y}_{\mathcal{R},n+l} = \sum_{i=1}^{n} \phi_{ni}^{(l)} Y_{\mathcal{R},n+1-i} = \boldsymbol{\phi}_n^{(l)} \mathbf{Y}_{\mathcal{R},n}, \tag{46}$$

where the vectors of weights $\boldsymbol{\phi}_n^{(l)} = (\phi_{n1}^{(l)}, \phi_{n2}^{(l)}, ..., \phi_{nn}^{(l)})'$ can be obtained from the analytical autocovariances of $Y_{\mathcal{R},t}$ at lags l and beyond. More precisely, $\boldsymbol{\phi}_n^{(l)}$ is any solution of $\boldsymbol{\Psi}_n \boldsymbol{\phi}_n^{(l)} = \boldsymbol{\kappa}_n^{(l)}$ where $\boldsymbol{\kappa}_n^{(l)} = (\kappa(l), \kappa(l+1), ..., \kappa(l+n-1))'$ denotes the autocovariances of $Y_{\mathcal{R},t}$ at lags l and beyond and $\boldsymbol{\Psi}_n = [\kappa(i-j)]_{i,j=1,\ldots,n}$ is the variance-covariance matrix. The autocovariances of $Y_{\mathcal{R},t}$ are based on the LMSM moments:

$$E\left[\theta_t^2\right] = \exp(2\lambda \cdot k), \tag{47}$$

and

$$E\left[\theta_{t+T}\theta_t\right] = \prod_{i=1}^{k}\left\{\left(1-(1-\gamma_i)^T\right)+(1-\gamma_i)^T\exp\left(2\cdot\lambda\right)\right\}. \tag{48}$$

B Data issues

The dataset displays time gaps overnight, during weekends and holidays. The overnight gaps hide a great amount of asset pricing information primarily originating from the East Asian stock markets, which enters the American and the European markets at once starting with the opening auctions. Visual inspection of the returns computed with the sample intervals of Table 1, indicates that the indices adjust similarly fast within at most 5 minutes after an overnight gap, weekends or holidays. Therefore, we cleaned the effects of time gaps from the data by removing returns of the first 5 minutes on every trading day.[7]

The estimation of RV models requires the use of equally long trading days. Thus, we removed all trading days with missing observations over a period greater or equal to 1.5 hours. Additionally, we deleted short trading days prior to holidays (Christmas, New Year, Thanksgiving Day and Independence Day). Some other data errors (simultaneous quotes, irregular quotes and erroneous timestamps) were fixed at this stage.

Missing observations during less than 1.5 hours were treated for the 30 minutes price series by means of interpolation (Dacorogna et al., 2001). This is the case when one or two consecutive 30-minute prices are missing. We employed three interpolation techniques in the following order of priority: previous-tick interpolation, next-tick interpolation and linear interpolation. Technical details can be provided upon request. Based on the clean 30 minutes price data, we constructed the 30 minutes returns, the daily returns y_t and daily realized volatility $\sigma^2_{\mathcal{R},t}$ to be employed in our study.

[7] The second by second quotation of DAX starting in 2006 allows for a more accurate evaluation of time gap effects. In this case we removed only the first 20 ticks on every trading day.

Figure 1: Daily realized variances $\sigma^2_{\mathcal{R},t}$

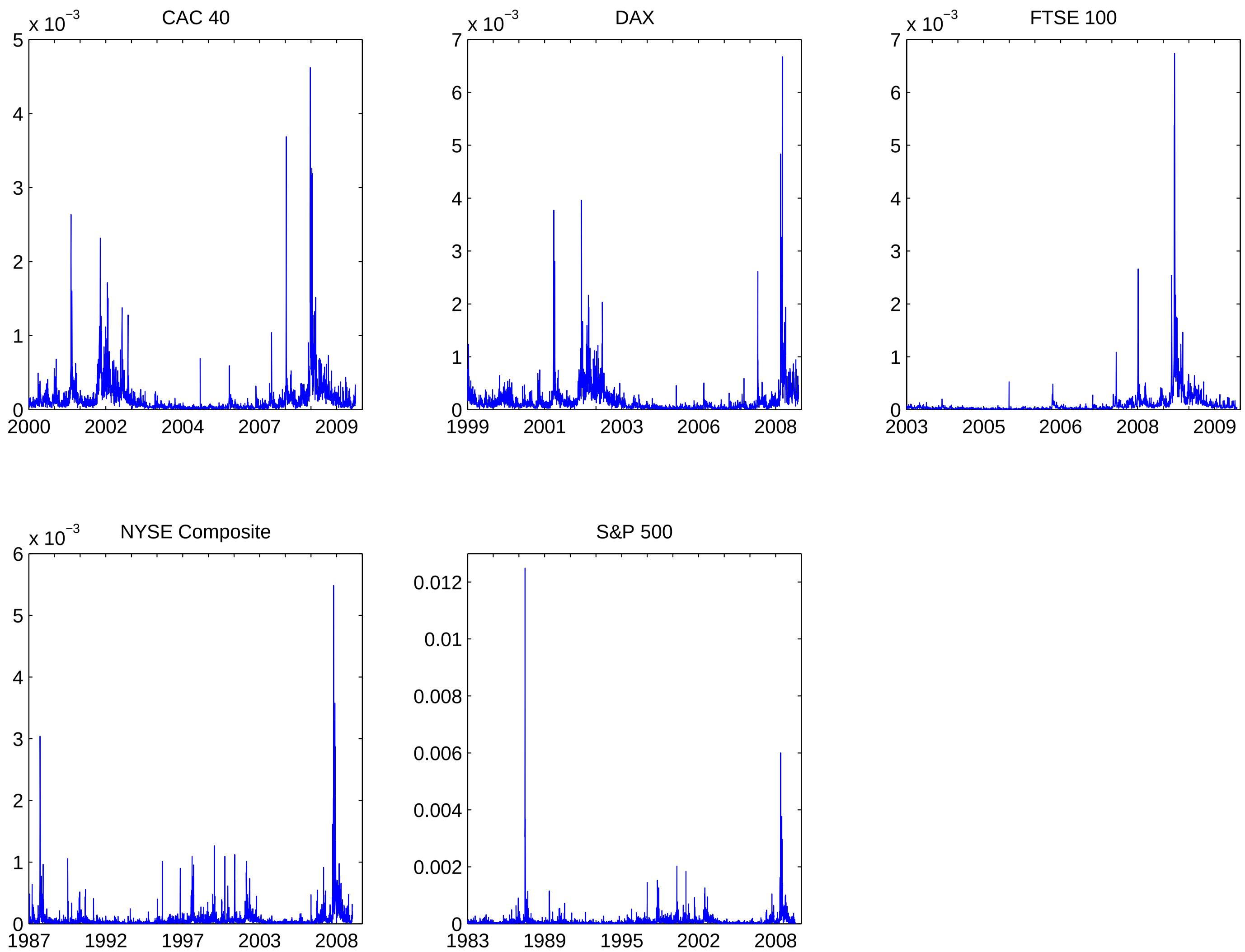


Figure 2: Daily realized returns y_t (top) and standardized returns $\tilde{y}_t$ (bottom)

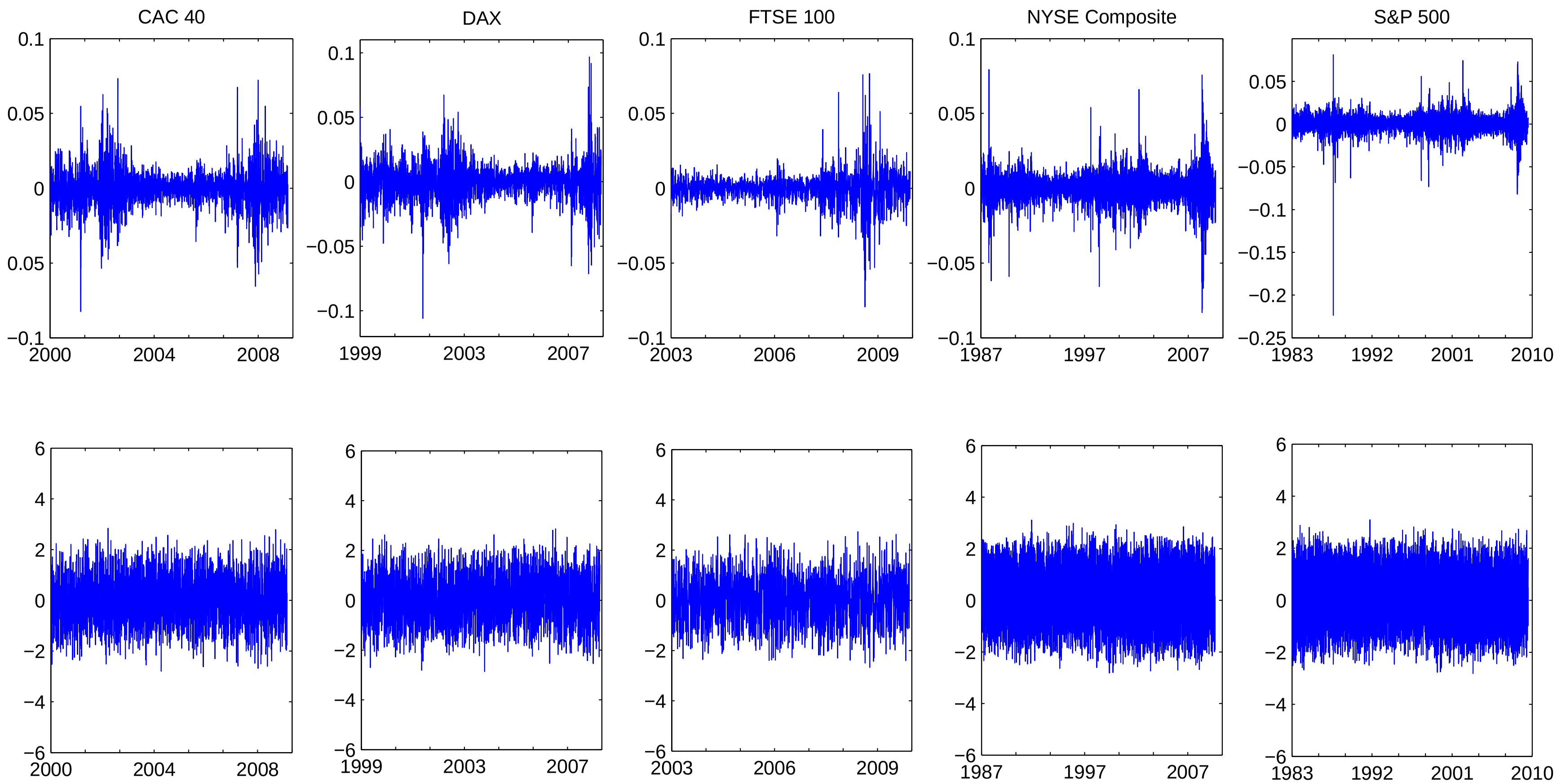


Figure 3: Kernel density estimates of daily returns standardized using its sample mean and its sample (thin solid line) or realized standard deviation (thick solid line). The dotted line is a standard normal distribution.

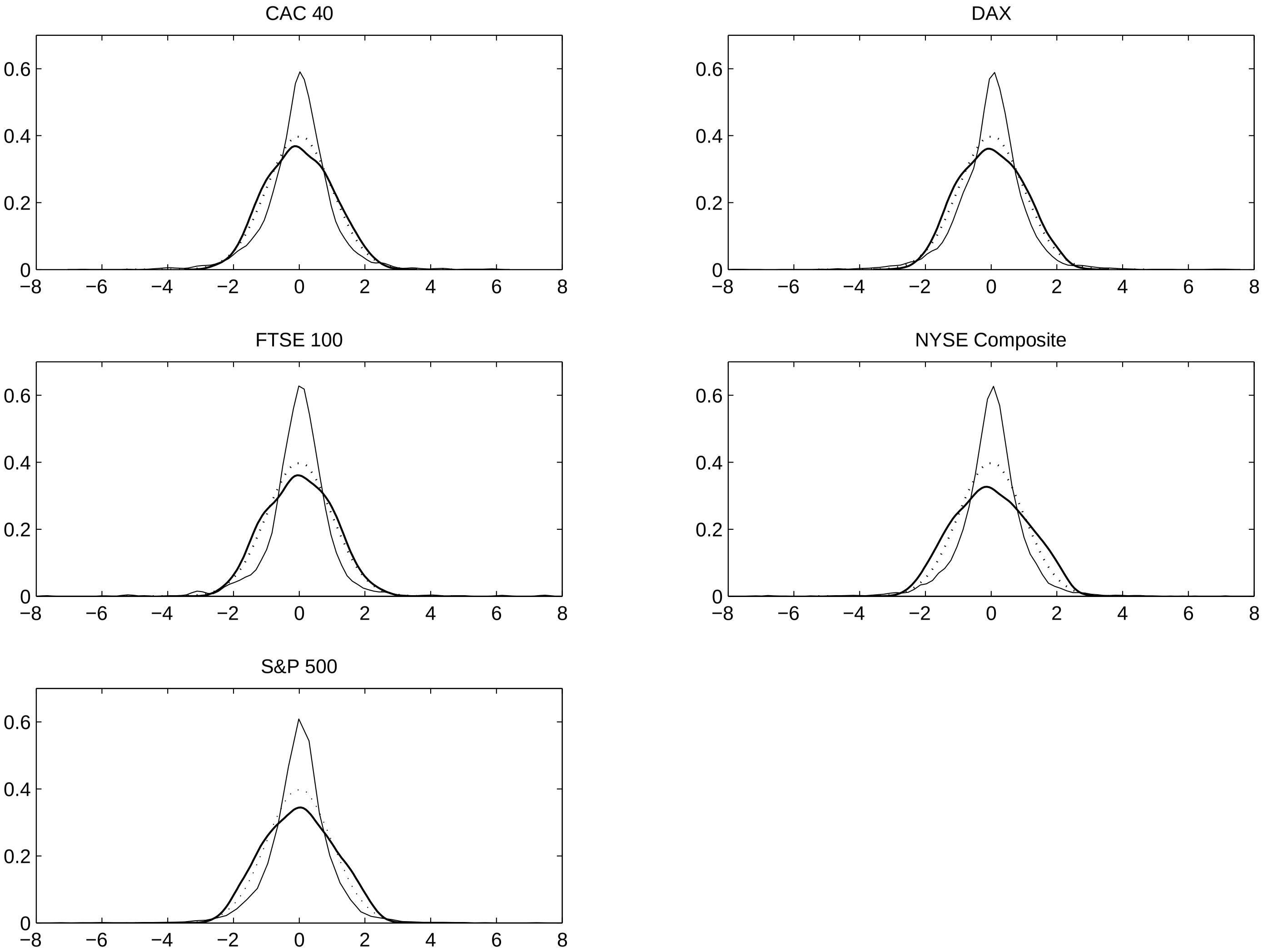
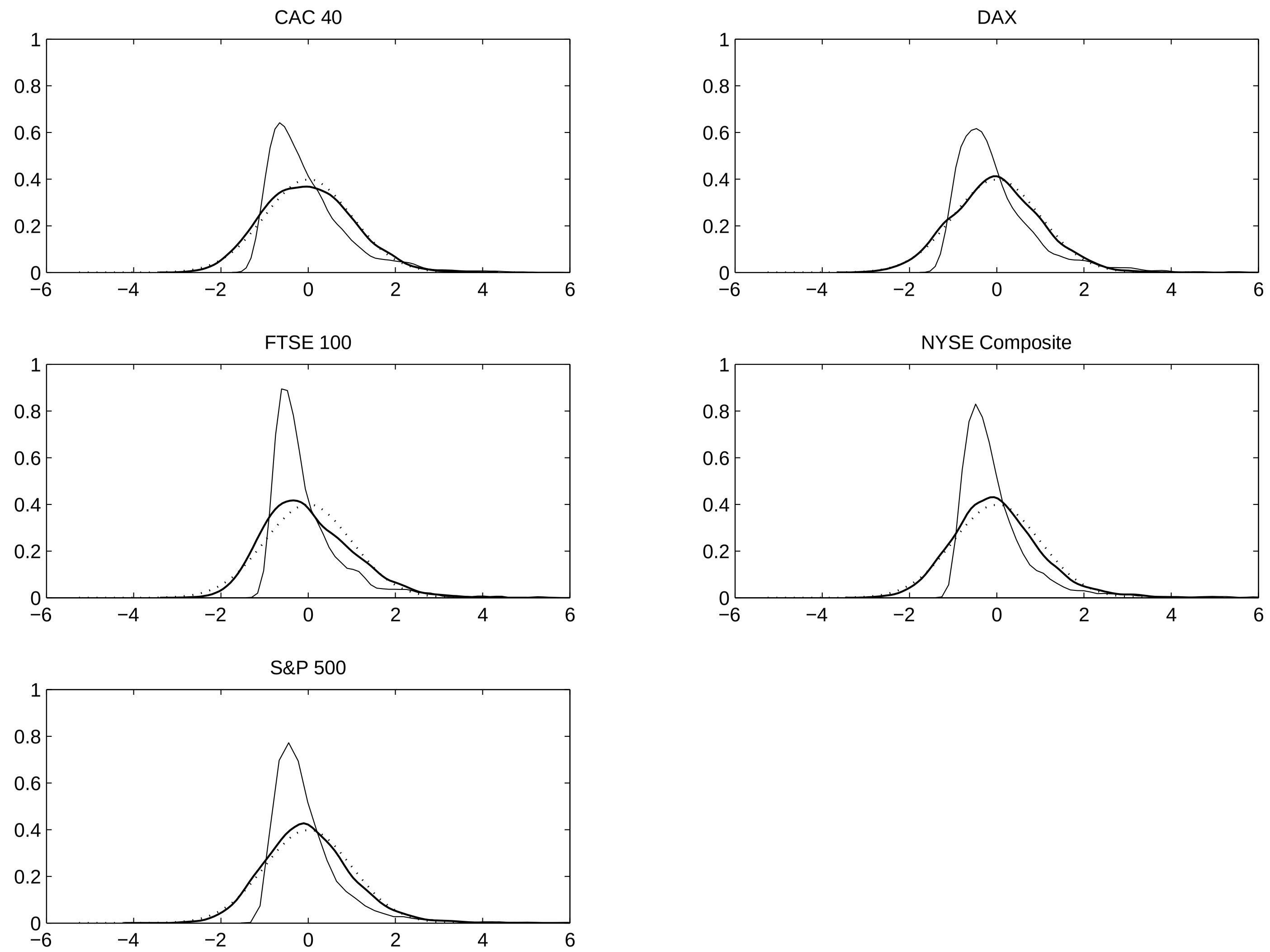


Figure 4: Kernel density estimates of the realized standard deviation (thin solid line) and of the realized logarithmic standard deviation (thick solid line) standardized to have zero mean and unit variance respectively. The dotted line is a standard normal distribution.



index	time span	trading time	sampling interval	sample size
CAC 40	13.06.2000 - 09.02.2010	9:00 - 17:30 CET	1 minute	1,274,950
DAX	04.01.1999 - 30.04.2009	9:00 - 17:30 CET	1 second	30,769,563
FTSE 100	01.07.2003 - 04.01.2010	9:00 - 17:30 CET	1 minute	836,593
NYSE Composite	02.01.1987 - 09.02.2010	9:30 - 16:00 EST	1 minute	2,122,469
S&P 500	01.02.1983 - 04.01.2010	9:30 - 16:00 EST	1 minute	2,554,292

Table 1: The dataset (CET = Central European Time, EST = Eastern Standard Time). Note: From 1999 to 2005 the DAX index was reported every 15 seconds, followed by second by second quotations since 2006. Before September 20, 1999 DAX was calculated during 8:30 - 17:00 CET. Before September 30, 1985 S&P 500 was calculated during 10:00 - 16:00 EST.

index	mean	std	skewness	kurtosis
returns				
CAC 40	-0.0006	0.0125	-0.0618	7.6014
DAX	-0.0007	0.0137	-0.1220	8.6528
FTSE 100	-0.0004	0.0103	0.0737	14.0491
NYSE Composite	0.00002	0.0092	-0.4021	13.3989
S&P 500	-0.00006	0.0100	-1.8300	46.5143
standardized returns				
CAC 40	-0.0094	0.9955	0.0532	2.5297
DAX	0.0124	0.9897	0.0449	2.4545
FTSE 100	-0.0103	1.0002	0.0203	2.5032
NYSE Composite	0.0894	1.1111	0.0388	2.3315
S&P 500	0.0479	1.0682	0.0378	2.3895
RV				
CAC 40	0.0104	0.0067	2.4069	13.1623
DAX	0.0115	0.0074	2.4133	13.3184
FTSE 100	0.0080	0.0063	3.8197	28.9645
NYSE Composite	0.0066	0.0049	4.0298	30.5654
S&P 500	0.0074	0.0053	5.0396	57.0249
logarithmic RV				
CAC 40	-4.7238	0.5536	0.2497	2.9202
DAX	-4.6311	0.5638	0.1670	3.0483
FTSE 100	-5.0162	0.5826	0.6241	3.4441
NYSE Composite	-5.1953	0.5546	0.5298	3.8464
S&P 500	-5.0577	0.5247	0.4851	3.8981

Table 2: Descriptive statistics for the index data

Acronym	Model
	Standard
M1	FIGARCH(1,d,1)
M2	GARCH(1,1)
M3	TGARCH(1,1)
M4	BMSM
M5	LMSM
M6	SV
	Realized
M7	RV-ARMA(p,0,q)
M8	RV-ARFIMA(1,d,1)
M9	RV-LMSM
M10	RV-BNS
	Combinations
C1	M2, M3, M5, M6
C2	M7, M8, M9, M10
C3	M2, M4, M9, M10

Table 3: Alternative models

	BMSM		LMSM		RV-LMSM	
	Estimation Sample until July 2005					
Market	$\hat{m}_0$	$\hat{\sigma}_{\mathcal{L}}$	$\hat{\lambda}$	$\hat{\sigma}_{\mathcal{L}}$	$\hat{\lambda}$	$\hat{\sigma}_{\mathcal{R}}$
DAX	1.192 (0.137)	1.193 (0.079)	0.021 (0.003)	1.194 (0.079)	0.050 (0.002)	1.632 (—)
FTSE 100	1.200 (0.053)	0.529 (0.014)	0.050 (0.004)	0.529 (0.029)	0.054 (0.003)	0.307 (—)
S&P 500	1.344 (0.048)	0.924 (0.066)	0.065 (0.002)	0.924 (0.066)	0.067 (0.003)	0.661 (—)
CAC 40	1.286 (0.122)	1.189 (0.079)	0.042 (0.004)	1.189 (0.079)	0.049 (0.002)	1.238 (—)
NYSE Composite	1.219 (0.087)	0.784 (0.035)	0.028 (0.002)	0.784 (0.035)	0.074 (0.003)	0.473 (—)
	Estimation Sample until July 2007					
Market	$\hat{m}_0$	$\hat{\sigma}_{\mathcal{L}}$	$\hat{\lambda}$	$\hat{\sigma}_{\mathcal{L}}$	$\hat{\lambda}$	$\hat{\sigma}_{\mathcal{R}}$
DAX	1.200 (0.315)	1.307 (0.090)	0.050 (0.003)	1.320 (0.089)	0.043 (0.003)	1.962 (—)
FTSE 100	1.204 (0.241)	0.513 (0.019)	0.021 (0.005)	0.513 (0.019)	0.038 (0.003)	0.287 (—)
S&P 500	1.323 (0.054)	0.952 (0.069)	0.058 (0.002)	0.952 (0.069)	0.068 (0.002)	0.693 (—)
CAC 40	1.368 (0.116)	1.232 (0.100)	0.075 (0.005)	1.232 (0.100)	0.044 (0.002)	1.543 (—)
NYSE Composite	1.162 (0.122)	0.809 (0.037)	0.020 (0.002)	0.809 (0.037)	0.070 (0.002)	0.496 (—)

Table 4: In-sample parameter estimates and standard deviations (in brackets) for the three multifractal models employed in this study

		Standard						Realized				Combinations		
		M1	M2	M3	M4	M5	M6	M7	M8	M9	M10	C1	C2	C3
	l	Sample: July 2007 - April 2009												
MSE	1	0.851	0.847	***0.838***	0.860	0.860	0.905	1.012	0.863	***0.837***	0.868	***0.831****	0.840	***0.832***
		0.691	0.670	***0.607***	0.694	0.691	0.787	1.013	0.613	0.636	***0.582****	0.666	0.632	0.613
	20	0.961	0.964	0.949	***0.945***	***0.945***	***0.945***	1.033	***0.891****	0.913	0.923	0.939	0.919	0.925
		0.902	0.911	0.909	***0.903***	***0.903***	0.905	1.052	***0.877****	0.889	0.895	0.888	0.893	0.884
	50	1.013	1.038	1.027	0.991	0.991	***0.986****	1.042	0.995	***0.989***	0.995	0.998	0.994	0.988
		0.989	1.016	1.012	0.975	0.975	***0.961****	1.066	1.000	***0.973***	0.990	0.976	0.986	0.971
	100	1.005	1.016	***0.997***	0.999	0.999	1.002	1.047	1.012	***0.994****	1.004	0.999	1.010	0.995
		1.004	1.024	***1.001***	1.003	1.002	1.012	1.076	1.059	***1.002***	1.007	1.002	1.026	***0.998****
MAE	1	1.094	1.096	1.135	***1.056***	***1.056***	1.166	***0.954****	1.094	1.061	1.101	1.094	1.018	1.069
		0.794	0.795	0.788	0.750	***0.749***	0.904	0.999	0.734	***0.721***	0.758	0.772	***0.693****	***0.711***
	20	1.127	1.153	1.177	***1.067***	***1.067***	1.151	***0.964****	1.078	1.065	1.043	1.122	1.019	1.067
		0.937	0.963	0.992	***0.890***	***0.890***	0.974	1.042	0.888	***0.870***	0.874	0.923	***0.845****	0.872
	50	1.118	1.170	1.174	***1.043***	***1.043***	1.099	***0.970****	1.073	1.056	1.012	1.104	1.011	1.046
		1.015	1.077	1.078	***0.950***	***0.950***	0.997	1.057	0.995	***0.951***	0.968	0.998	***0.931****	0.943
	100	1.040	1.074	1.051	***0.999***	***0.999***	1.014	***0.986****	1.021	1.005	0.996	1.023	0.988	1.001
		1.003	1.042	1.011	***0.985***	***0.985***	0.993	1.096	1.070	***0.995***	1.005	0.993	1.008	***0.981****
	l	Sample: July 2005 - July 2007												
MSE	1	***0.347***	0.349	***0.347***	0.350	0.351	0.361	0.393	***0.335****	0.343	0.348	0.340	0.344	0.341
		0.104	0.103	0.104	***0.102****	***0.102****	0.138	0.183	***0.108***	0.119	0.109	***0.102****	0.115	***0.102****
	20	***0.371***	0.389	0.393	0.381	0.385	0.383	0.412	***0.356****	0.394	0.590	0.375	0.404	0.400
		0.152	0.176	0.197	0.175	0.180	0.174	0.212	***0.148****	0.193	0.456	0.164	0.205	0.202
	50	***0.400***	0.455	0.469	0.427	0.433	0.409	0.436	***0.386****	0.454	0.829	0.416	0.459	0.468
		0.170*	0.245	0.277	0.219	0.226	0.196	0.231	***0.172***	0.254	0.766	0.202	0.263	0.274
	100	***0.442***	0.593	0.582	0.486	0.493	0.453	0.472	***0.431****	0.530	0.907	0.475	0.506	0.528
		0.192*	0.400	0.400	0.274	0.285	0.225	0.253	***0.201***	0.336	0.868	0.250	0.301	0.334
MAE	1	***0.341***	0.356	0.363	0.385	0.387	0.409	0.478	***0.334****	0.393	0.388	0.371	0.392	0.375
		0.179	0.191	0.198	0.219	0.221	0.258	0.365	***0.173****	0.242	0.228	0.202	0.239	0.208
	20	***0.357***	0.402	0.425	0.439	0.446	0.420	0.495	***0.341****	0.466	0.696	0.409	0.484	0.477
		0.229	0.281	0.307	0.324	0.334	0.296	0.392	***0.214****	0.359	0.647	0.283	0.379	0.372
	50	***0.382***	0.487	0.520	0.495	0.505	0.447	0.515	***0.347****	0.536	0.887	0.459	0.547	0.556
		0.263	0.376	0.416	0.388	0.400	0.329	0.406	***0.241****	0.438	0.864	0.343	0.449	0.462
	100	***0.406***	0.632	0.641	0.555	0.566	0.481	0.536	***0.352****	0.612	0.939	0.518	0.582	0.610
		0.289	0.544	0.558	0.455	0.469	0.367	0.430	***0.255****	0.525	0.925	0.410	0.488	0.522

	l	Sample: July 2005 - April 2009												
MSE	1	0.833	0.832	***0.826***	0.845	0.845	0.899	0.979	0.849	***0.824***	0.851	***0.813*** *	0.826	0.818
		0.671	0.649	***0.593***	0.667	0.664	0.754	0.966	0.607	0.625	***0.568*** *	0.639	0.619	0.596
	20	0.950	0.968	0.954	***0.932***	***0.932***	0.936	0.998	***0.881*** *	0.899	0.917	0.927	0.903	0.912
		0.890	0.913	0.916	***0.886***	0.887	0.890	1.001	***0.861*** *	0.869	0.885	0.872	0.870	0.868
	50	1.011	1.074	1.051	0.979	0.978	***0.973*** *	1.014	0.988	***0.975***	0.991	0.992	0.976	0.976
		0.989	1.060	1.046	0.961	0.961	***0.940*** *	1.027	0.989	***0.955***	0.984	0.967	0.962	0.956
	100	1.015	1.045	1.005	***0.991***	***0.991***	0.984	1.024	1.008	***0.981*** *	1.000	0.992	0.992	0.986
		1.023	1.071	1.018	***0.993***	***0.993***	0.991	1.045	1.051	***0.983*** *	1.001	0.996	1.001	0.988
MAE	1	0.798	0.819	0.842	***0.795***	0.797	0.909	***0.765*** *	0.795	0.803	0.824	0.820	0.775	0.799
		0.553	0.568	0.567	***0.543***	***0.543***	0.686	0.734	***0.514*** *	0.537	0.552	0.557	0.518	0.515
	20	0.827	0.893	0.910	***0.819***	0.819	0.896	***0.780*** *	0.795	0.837	0.907	0.855	0.813	0.837
		0.664	0.732	0.760	0.670	0.673	0.735	0.766	***0.629*** *	0.677	0.791	0.685	0.665	0.680
	50	0.833	0.970	0.964	***0.827***	0.828	0.868	***0.794*** *	0.803	0.860	0.961	0.867	0.835	0.856
		0.744	0.885	0.883	***0.738***	0.741	0.764	0.791	***0.724*** *	0.763	0.931	0.768	0.749	0.764
	100	0.792	0.978	0.922	0.823	0.827	***0.816***	0.810	***0.772*** *	0.855	0.973	0.833	0.833	0.848
		0.764	0.935	0.881	0.786	0.791	***0.761*** *	0.826	***0.778***	0.815	0.971	0.783	0.804	0.809

Table 5: MSE and MAE for CAC 40. The table displays the results of the relative MSE and MAE for each of the forecasting horizons $l = 1, 20, 50, 100$ measured against squared returns (above) and realized volatility (below). Models - M1: FIGARCH - M2: GARCH - M3: TGARCH - M4: BMSM - M5: LMSM - M6: SV - M7: RV-ARMA - M8: RV-ARFIMA - M9: RV-LMSM - M10: RV-BNS. Combinations - C1: M2, M3, M5, M6 - C2: M7, M8, M9, M10 - C3: M2, M4, M9, M10. Bold italics indicate the best models from groups M1 to M6 and M7 to M10, respectively, for each time horizon and criterion. Bold italics in the categories of combined models (C1 to C3) stand for improvements against all single models. The best forecast for each time horizon and criterion is also indicated by an asterisk.

		Standard						Realized				Combinations		
		M1	M2	M3	M4	M5	M6	M7	M8	M9	M10	C1	C2	C3
	l	Sample: July 2007 - April 2009												
MSE	1	0.848	0.847	***0.825***	0.876	0.874	0.910	0.992	0.802	0.807	***0.769****	0.848	0.810	0.810
		0.735	0.701	***0.627***	0.719	0.712	0.800	0.979	0.590	0.622	***0.548****	0.692	0.610	0.616
	20	0.967	0.980	***0.955***	0.967	0.967	0.958	1.012	***0.941****	0.942	0.948	0.955	0.949	0.950
		0.972	1.013	0.949	0.943	0.942	***0.925***	1.027	0.916	***0.911****	0.914	0.934	0.917	0.917
	50	1.018	1.044	1.015	1.004	1.005	***0.995****	1.020	1.014	***0.995****	0.996	1.003	0.999	0.997
		1.055	1.111	1.037	1.010	1.011	***0.984****	1.041	1.024	0.992	***0.992***	1.014	0.998	0.995
	100	1.015	1.025	1.017	1.013	1.014	***1.011***	1.025	1.045	1.004	***1.000****	1.011	1.015	1.006
		1.055	1.085	1.047	1.034	1.035	***1.029***	1.052	1.097	1.016	***1.000****	1.037	1.034	1.018
MAE	1	1.069	1.073	1.050	***1.023***	1.024	1.059	***0.938****	1.000	0.993	0.979	1.044	0.956	1.009
		0.794	0.794	***0.718***	0.726	0.723	0.818	0.929	***0.676***	0.686	0.682	0.735	***0.656****	0.685
	20	1.111	1.145	1.082	1.042	1.045	***1.025***	***0.946****	0.999	1.032	1.014	1.062	0.982	1.040
		0.955	1.011	0.930	0.893	0.893	***0.879***	0.971	***0.836***	0.855	0.876	0.901	***0.829****	0.867
	50	1.101	1.159	1.052	1.018	1.020	***0.999***	***0.957****	0.977	1.022	0.999	1.038	0.972	1.020
		1.081	1.169	1.021	0.973	0.976	***0.941***	0.997	0.963	***0.961***	0.977	0.995	***0.923****	0.963
	100	1.056	1.105	1.005	0.994	0.995	***0.983***	***0.975****	0.985	1.006	1.000	1.008	0.981	1.002
		1.077	1.134	1.034	1.017	1.019	***0.995****	1.035	1.083	1.005	***0.999***	1.026	0.999	1.007
	l	Sample: July 2005 - July 2007												
MSE	1	0.409	0.415	***0.406***	0.412	0.411	0.440	0.487	***0.405****	0.408	0.407	0.443	0.408	0.438
		0.100	0.100	***0.100***	0.160	0.170	0.145	0.240	***0.084****	0.111	0.099	0.164	0.107	0.167
	20	***0.444***	0.463	0.452	0.476	0.478	0.480	0.512	***0.442****	0.473	0.728	0.483	0.478	0.548
		0.126	0.159	0.156	0.241	0.250	0.193	0.277	***0.119****	0.210	0.605	0.210	0.221	0.331
	50	***0.451****	0.495	0.475	0.555	0.557	0.492	0.528	***0.456***	0.545	0.962	0.506	0.523	0.616
		0.139	0.216	0.198	0.341	0.344	0.221	0.302	***0.138****	0.321	0.945	0.245	0.289	0.431
	100	***0.503****	0.612	0.558	0.676	0.675	0.544	0.580	***0.519***	0.664	1.007	0.564	0.575	0.676
		0.162*	0.345	0.277	0.482	0.481	0.253	0.341	***0.168***	0.478	1.010	0.291	0.335	0.497
MAE	1	***0.403***	0.418	0.411	0.457	0.456	0.465	0.586	***0.377****	0.452	0.439	0.515	0.452	0.520
		0.210	0.220	0.214	0.231	0.230	0.287	0.457	***0.183****	0.276	0.250	0.357	0.273	0.366
	20	***0.419***	0.457	0.453	0.554	0.557	0.485	0.602	***0.373****	0.547	0.810	0.540	0.557	0.646
		0.255	0.305	0.301	0.421	0.423	0.338	0.486	***0.211****	0.416	0.762	0.405	0.428	0.545
	50	***0.434***	0.519	0.510	0.640	0.641	0.504	0.619	***0.364****	0.635	0.976	0.568	0.611	0.710
		0.281	0.378	0.369	0.541	0.543	0.371	0.511	***0.233****	0.532	0.970	0.441	0.502	0.629
	100	***0.465***	0.631	0.593	0.754	0.756	0.545	0.652	***0.384****	0.743	1.004	0.611	0.651	0.755
		0.308	0.510	0.466	0.701	0.705	0.419	0.546	***0.252****	0.662	1.006	0.489	0.543	0.677

	l	Sample: July 2005 - April 2009												
MSE	1	0.841	0.840	***0.819***	0.866	0.863	0.910	0.970	0.806	0.801	***0.762****	0.841	0.806	0.803
		0.720	0.689	***0.620***	0.691	0.657	0.785	0.924	0.593	0.604	***0.527****	0.668	0.597	0.596
	20	0.958	0.986	***0.953***	0.957	0.957	0.953	0.988	0.953	***0.932****	0.947	0.948	0.944	0.941
		0.938	1.018	0.939	0.910	***0.908***	0.900	0.969	0.909	***0.881****	0.906	0.907	0.893	0.888
	50	1.011	1.071	1.018	0.994	0.994	***0.983****	1.001	1.017	***0.985***	0.996	0.996	0.989	0.988
		1.026	1.163	1.037	0.979	0.979	***0.951****	0.994	1.019	***0.963***	0.992	0.989	0.969	0.968
	100	1.017	1.057	1.011	1.003	1.003	***1.000***	1.009	1.053	***0.994****	1.000	1.004	1.004	0.998
		1.047	1.157	1.030	1.005	1.004	***0.995***	1.011	1.098	***0.990****	1.000	1.014	1.003	0.995
MAE	1	0.816	0.837	0.819	***0.808***	0.813	0.845	0.804	***0.754****	0.793	0.777	0.817	0.766	0.793
		0.561	0.578	0.531	0.538	***0.526***	0.617	0.726	***0.473****	0.526	0.509	0.536	0.504	0.508
	20	0.844	0.915	0.874	0.836	0.841	***0.832***	0.816	***0.741****	0.855	0.940	0.845	0.819	0.856
		0.667	0.763	0.703	0.670	0.673	0.673	0.758	***0.588****	0.683	0.839	0.668	0.671	0.689
	50	0.838	0.973	0.892	0.832	0.838	***0.822***	0.829	***0.730****	0.880	0.991	0.843	0.833	0.867
		0.753	0.926	0.811	0.739	0.746	***0.721***	0.787	***0.678****	0.793	0.979	0.749	0.752	0.780
	100	0.816	0.974	0.876	0.830	0.838	***0.812***	0.851	***0.754****	0.902	1.002	0.833	0.850	0.870
		0.771	0.954	0.829	0.781	0.790	***0.748****	0.821	***0.781***	0.860	1.002	0.778	0.807	0.826

Table 6: MSE, MAE for DAX. For further notes see Table 5. Models - M1: FIGARCH - M2: GARCH - M3: TGARCH - M4: BMSM - M5: LMSM - M6: SV - M7: RV-ARMA - M8: RV-ARFIMA - M9: RV-LMSM - M10: RV-BNS. Combinations - C1: M2, M3, M5, M6 - C2: M7, M8, M9, M10 - C3: M2, M4, M9, M10.

		Standard						Realized				Combinations		
		M1	M2	M3	M4	M5	M6	M7	M8	M9	M10	C1	C2	C3
	l	Sample: July 2007 - April 2009												
	1	0.805	0.813	0.807	0.798	***0.791***	0.980	0.982	0.770	0.753	***0.734****	0.823	0.778	0.761
		0.672	0.664	0.662	0.649	***0.623***	0.966	0.970	0.625	0.598	***0.566****	0.691	0.641	0.606
	20	0.898	0.908	0.911	***0.891***	0.892	0.999	0.997	***0.877****	0.888	0.972	0.912	0.929	0.901
		0.841	0.859	0.871	0.831	***0.830****	0.998	0.995	0.867	***0.833***	0.955	0.864	0.891	0.847
MSE														
	50	0.943	0.969	0.972	***0.938***	0.939	0.999	1.001	***0.922****	0.941	0.999	0.956	0.966	0.947
		0.908	0.949	0.953	***0.899****	0.901	0.999	1.002	0.930	***0.903***	0.999	0.928	0.945	0.913
	100	0.963	0.993	0.993	***0.953***	0.954	0.999	1.002	***0.948****	0.955	1.000	0.974	0.981	0.964
		0.944	0.990	0.989	***0.929****	0.931	0.999	1.003	0.971	***0.933***	1.000	0.960	0.971	0.946
	1	1.023	0.996	***0.965***	1.035	1.039	0.981	***0.983***	0.987	1.015	0.964	0.964	***0.944****	0.999
		0.625	0.601	***0.571****	0.602	0.601	0.944	0.955	***0.578***	0.594	0.596	0.600	0.594	0.586
	20	1.023	0.996	***0.995***	1.060	1.059	0.999	0.997	0.974*	1.063	***0.984***	0.992	0.977	1.013
		0.721	0.747	0.746	0.734	0.734	0.996	0.990	0.742	***0.732***	0.917	0.737	0.780	***0.719****
MAE														
	50	1.020	0.997	***0.992***	1.061	1.060	0.999	1.001	***0.974****	1.056	0.999	1.000	0.985	1.015
		0.802*	0.874	0.880	0.818	0.818	0.998	1.005	0.834	***0.820***	0.997	0.824	0.860	0.805
	100	***0.984***	0.993	0.993	1.000	1.000	0.999	1.002	***0.980****	0.997	1.000	0.983	0.983	0.986
		0.866	0.968	0.966	***0.844****	0.845	0.998	1.008	0.926	***0.854***	1.000	0.895	0.923	0.868
	l	Sample: July 2005 - July 2007												
	1	0.956	1.005	0.994	0.926	***0.922***	1.001	0.986	0.916	0.913	***0.908****	0.952	0.923	0.918
		0.949	0.955	0.945	0.728	***0.664****	1.000	0.948	0.789	***0.748***	0.758	0.821	0.785	0.780
	20	1.012	1.041	1.055	0.997	***0.994***	1.007	0.995	***0.953****	1.010	0.994	1.005	1.003	1.000
		1.036	1.091	1.187	1.036	1.042	***1.020***	***0.989****	1.086	1.034	1.002	1.021	1.013	1.019
MSE														
	50	1.037	1.075	1.122	1.021	1.018	***1.009***	0.998	***0.993****	1.029	0.999	1.031	1.019	1.017
		1.075	1.156	1.332	1.050	1.048	***1.026***	***0.992****	1.204	1.064	0.996	1.056	1.044	1.036
	100	1.045	1.086	1.200	1.025	1.023	***1.009***	***0.998****	1.014	1.034	***0.998****	1.042	1.025	1.021
		1.098	1.176	1.506	1.046	1.047	***1.027***	0.997	1.321	1.077	***0.994****	1.070	1.068	1.043
	1	1.001	0.999	1.128	1.030	1.030	***0.980****	1.005	***0.982***	0.997	1.004	1.019	0.993	1.002
		0.920	0.982	1.114	0.888	***0.876***	0.971	0.976	0.845	***0.840****	0.863	0.904	0.858	0.863
	20	0.990	1.024	1.174	1.050	1.048	***0.981***	1.007	***0.964****	1.018	1.023	1.032	0.999	1.016
		*0.977**	1.078	1.301	1.055	1.058	***0.985***	0.998	***0.990***	1.001	1.007	1.025	***0.977****	0.998
MAE														
	50	0.989	1.040	1.252	1.066	1.065	***0.983***	1.010	***0.950****	1.017	1.013	1.051	0.990	1.017
		1.023	1.174	1.471	1.117	1.114	***0.995****	***1.002***	1.117	1.044	1.010	1.100	1.006	1.033
	100	***0.982***	1.041	1.353	1.058	1.055	0.984	1.007	***0.951****	1.003	1.008	1.063	0.979	1.005
		1.037	1.207	1.628	1.110	1.107	***0.999****	1.005	1.255	1.055	***1.007***	1.132	1.018	1.035

	l	Sample: July 2005 - April 2009												
	1	0.863	0.854	0.852	0.797	***0.796***	0.997	0.991	0.780	***0.758****	0.773	0.838	0.799	0.786
		0.795	0.764	0.760	0.645	***0.643***	0.995	0.986	0.637	***0.607****	0.624	0.735	0.671	0.649
	20	0.901	0.902	0.931	***0.890***	***0.890***	1.001	0.994	***0.863****	0.887	0.908	0.895	0.909	0.888
		0.853	0.848	0.906	***0.828****	***0.828****	1.001	0.991	0.845	***0.830***	0.863	0.839	0.863	0.830
MSE														
	50	0.955	0.938	1.001	***0.937***	***0.937***	1.002	0.996	***0.923****	0.940	0.974	0.938	0.960	0.940
		0.925	0.899	1.003	***0.897****	***0.897****	1.003	0.994	0.930	***0.900***	0.957	0.898	0.934	0.901
	100	0.954	***0.951***	0.983	0.952	0.952	1.002	0.998	0.958	***0.953***	0.994	***0.946****	0.980	0.955
		0.932	***0.923***	0.984	0.926	0.927	1.004	0.996	0.987	***0.929***	0.992	***0.917****	0.970	0.931
	1	1.127	1.035	1.195	1.031	1.032	***0.995***	0.992	1.000	1.012	***0.977***	1.028	***0.957****	1.026
		0.793	0.707	0.838	***0.616***	***0.616***	0.992	0.975	***0.600****	0.608	0.607	0.672	0.612	0.628
	20	1.152	1.062	1.279	1.055	1.056	***0.999***	0.995	***0.985****	1.058	0.998	1.067	0.991	1.053
		0.846	0.780	0.997	0.751	0.751	1.003	0.981	***0.736****	0.746	0.764	0.767	0.752	0.745
MAE														
	50	1.164	1.060	1.328	1.056	1.056	***1.001***	0.997	***0.979****	1.050	0.996	1.086	0.992	1.049
		0.950	0.853	1.123	***0.835***	***0.836***	1.008	0.986	0.858	***0.834***	0.907	0.863	0.853	***0.833****
	100	1.078	1.005	1.247	1.003	1.003	***1.001***	0.998	***0.987***	0.997	0.995	1.036	0.984*	1.001
		0.913	0.873	1.035	***0.860****	***0.860****	1.009	0.991	0.982	***0.865***	0.978	0.872	0.928	0.867

Table 7: MSE and MAE for FTSE 100. For further notes see Table 5. Models - M1: FIGARCH - M2: GARCH - M3: TGARCH - M4: BMSM - M5: LMSM - M6: SV - M7: RV-ARMA - M8: RV-ARFIMA - M9: RV-LMSM - M10: RV-BNS. Combinations - C1: M2, M3, M5, M6 - C2: M7, M8, M9, M10 - C3: M2, M4, M9, M10.

		Standard						Realized				Combinations		
		M1	M2	M3	M4	M5	M6	M7	M8	M9	M10	C1	C2	C3
	l	Sample: July 2007 - April 2009												
MSE	1	0.692	0.686	***0.650****	0.707	0.702	0.793	1.003	0.690	0.686	***0.676***	0.695	0.732	0.681
		0.438	0.458	***0.389****	0.472	0.461	0.651	1.005	***0.489***	0.499	0.503	0.461	0.557	0.455
	20	0.858	0.830	***0.823****	0.851	0.849	0.889	1.016	0.904	***0.868***	1.010	0.849	0.936	0.877
		0.773	0.740	***0.727****	0.775	0.772	0.825	1.028	0.842	***0.788***	1.017	0.764	0.889	0.798
	50	***0.952****	0.960	0.962	0.954	0.954	0.956	1.020	0.975	***0.964***	1.012	0.953	0.986	0.959
		0.928*	0.947	0.944	0.945	0.945	0.934	1.035	0.961	***0.951***	1.020	0.932	0.976	0.938
	100	***0.976****	0.984	0.992	0.978	0.978	0.992	1.022	1.003	***0.992***	1.012	0.982	1.005	0.985
		0.964*	0.979	0.990	0.978	0.979	0.990	1.037	1.009	***0.993***	1.020	0.976	1.009	0.980
MAE	1	0.975	0.987	***0.961***	0.979	0.978	1.028	0.996	0.929	0.932	***0.900****	0.978	0.913	0.937
		0.692	0.724	***0.666***	0.717	0.711	0.821	1.014	0.628	0.641	***0.628***	0.703	0.633	0.626*
	20	1.028	1.040	***1.012***	1.045	1.045	1.044	1.008	***0.998***	1.010	1.004	1.028	***0.978****	0.999
		0.872	0.865	***0.835****	0.899	0.898	0.912	1.041	0.855	***0.854***	1.022	0.865	0.873	0.838
	50	1.034	1.052	***1.021***	1.067	1.068	***1.021***	1.011	***1.000***	1.025	1.006	1.031	***0.994****	1.011
		0.945	0.956	***0.926***	1.000	1.001	0.928	1.046	***0.903****	0.935	1.024	0.937	0.922	0.913
	100	1.023	1.021	***1.006****	1.042	1.042	1.015	1.017	1.012	1.029	***1.008***	1.019	1.008	1.018
		0.966	0.965	***0.961****	1.002	1.002	0.967	1.053	0.991	***0.989***	1.026	0.965	0.990	0.967
	l	Sample: July 2005 - July 2007												
MSE	1	0.832	0.823	***0.810***	0.817	0.816	0.831	0.816	***0.803****	0.806	0.821	0.810	***0.803****	0.819
		0.431	0.429	0.429	0.415	***0.414***	0.421	0.400	***0.345****	***0.345****	0.399	0.403	0.353	0.449
	20	0.878	0.881	0.867	0.855	***0.854***	0.858	***0.834****	0.862	0.847	0.876	0.853	0.836	0.869
		0.595	0.604	0.602	0.516	***0.515***	0.475	0.434	0.426	***0.424****	0.623	0.527	0.434	0.597
	50	0.918	0.945	0.912	0.877	0.876	***0.872***	***0.843****	0.883	0.855	0.880	0.882	0.847	0.886
		0.700	0.775	0.708	0.571	0.570	***0.507***	0.451	0.457	0.450*	0.633	0.599	0.457	0.640
	100	0.943	0.987	0.924	0.883	0.882	***0.870***	***0.848****	0.883	0.855	0.885	0.892	0.851	0.896
		0.790	0.919	0.764	0.607	0.606	***0.530***	0.471	***0.465****	***0.465****	0.645	0.650	0.473	0.675
MAE	1	0.704	0.690	0.695	0.687	0.686	***0.652***	0.654	***0.586****	0.597	0.655	0.682	0.617	0.719
		0.488	0.470	0.477	0.473	0.472	***0.431***	0.439	***0.327****	0.337	0.423	0.456	0.367	0.535
	20	0.793	0.780	0.776	0.731	0.730	***0.673***	0.671	***0.602****	0.637	0.822	0.737	0.670	0.802
		0.665	0.641	0.646	0.570	0.569	***0.481***	0.473	***0.378****	0.420	0.713	0.583	0.469	0.683
	50	0.856	0.884	0.847	0.765	0.764	***0.692***	0.679	***0.607****	0.652	0.823	0.783	0.678	0.824
		0.762	0.803	0.758	0.625	0.624	***0.515***	0.486	***0.398****	0.451	0.716	0.656	0.487	0.718
	100	0.907	0.965	0.886	0.794	0.793	***0.718***	0.677	***0.593****	0.657	0.824	0.822	0.677	0.841
		0.841	0.934	0.820	0.669	0.668	***0.556***	0.497	***0.389****	0.470	0.721	0.713	0.497	0.746

	l	Sample: July 2005 - April 2009												
	1	0.689	0.685	***0.650****	0.733	0.728	0.797	1.003	***0.687***	0.688	0.680	0.700	0.732	0.688
		0.433	0.459	***0.391****	0.522	0.511	0.657	1.003	***0.484***	0.499	0.507	0.471	0.555	0.466
	20	0.860	0.830	***0.821****	0.863	0.861	0.891	1.016	0.896	***0.870***	1.010	0.852	0.933	0.883
		0.774	0.744	***0.728****	0.788	0.785	0.828	1.026	0.828	***0.789***	1.015	0.767	0.883	0.804
MSE														
	50	***0.954****	0.964	0.962	0.955	0.956	0.957	1.020	0.974	***0.966***	1.011	***0.954****	0.986	0.961
		0.930*	0.957	0.946	***0.942***	***0.943***	0.936	1.034	0.958	***0.952***	1.018	***0.934***	0.975	0.940
	100	***0.979****	0.983	0.990	***0.979****	***0.979****	0.992	1.022	1.004	***0.994***	1.011	0.983	1.006	0.986
		0.967*	0.980	0.987	0.976	0.977	0.989	1.037	1.010	***0.994***	1.019	0.977	1.009	0.981
	1	0.934	0.949	***0.926***	0.938	0.937	0.976	0.944	***0.878***	0.881	0.863*	0.936	0.868	0.894
		0.659	0.696	***0.647***	0.696	0.693	0.770	0.924	***0.583****	0.594	0.598	0.672	0.591	0.598
	20	0.994	1.010	***0.984***	0.994	0.995	0.995	0.957	***0.943***	0.954	0.976	0.987	***0.932****	0.960
		0.843	0.845	***0.818***	0.851	0.851	0.859	0.954	***0.784****	0.790	0.974	0.829	0.810	0.802
MAE														
	50	***1.009***	1.043	1.005	1.013	1.016	0.979	0.963	***0.947****	0.972	0.979	0.998	0.950	0.976
		0.925	0.960	***0.915***	0.939	0.942	0.885	0.967	***0.840****	0.872	0.979	0.906	0.863	0.876
	100	1.006	1.019	0.991	1.001	1.002	***0.975***	0.969	***0.956****	0.976	0.981	0.992	0.963	0.985
		0.953	0.973	0.942	0.952	0.954	***0.919***	0.980	***0.918****	0.926	0.984	0.936	0.927	0.930

Table 8: MSE and MAE for NYSE Composite. For further notes see Table 5. Models - M1: FIGARCH - M2: GARCH - M3: TGARCH - M4: BMSM - M5: LMSM - M6: SV - M7: RV-ARMA - M8: RV-ARFIMA - M9: RV-LMSM - M10: RV-BNS. Combinations - C1: M2, M3, M5, M6 - C2: M7, M8, M9, M10 - C3: M2, M4, M9, M10.

		Standard						Realized				Combinations		
		M1	M2	M3	M4	M5	M6	M7	M8	M9	M10	C1	C2	C3
	l	Sample: July 2007 - April 2009												
MSE	1	0.708	0.695	***0.650****	0.695	0.693	0.835	1.010	0.886	0.682	***0.679***	0.703	0.766	0.670
		0.448	0.480	***0.405****	0.467	0.461	0.739	1.014	0.825	***0.524***	0.564	0.483	0.652	0.467
	20	0.864	0.844	***0.835****	0.862	0.863	0.916	1.028	0.950	***0.878***	1.015	0.862	0.947	0.884
		0.799	0.771	***0.754****	0.802	0.803	0.880	1.041	0.950	***0.822***	1.022	0.797	0.925	0.827
	50	***0.961****	0.980	0.970	0.965	0.965	0.963	1.029	0.991	***0.976***	1.015	0.962	0.990	0.968
		0.951*	0.989	0.965	0.961	0.959	***0.951****	1.044	***0.963***	0.976	1.021	0.952	0.980	0.959
	100	***0.982****	0.985	0.990	0.989	0.989	0.996	1.031	***0.991***	1.004	1.015	0.988	1.003	0.993
		0.977	0.990	0.990	0.992	0.991	0.997	1.045	***0.959****	1.009	1.021	0.986	0.997	0.993
MAE	1	1.011	1.030	***0.979***	0.998	0.996	1.030	0.990	***0.953***	0.972	0.940	0.999	***0.936****	0.968
		0.730	0.773	***0.685***	0.722	0.716	0.853	1.022	0.801	***0.693***	0.694	0.721	0.704	***0.656****
	20	1.057	1.095	***1.045***	1.053	1.050	1.055	1.003	1.034	1.033	***1.001***	1.046	***0.980****	1.014
		0.917	0.949	***0.887***	0.918	0.915	0.952	1.057	1.054	***0.899***	1.025	0.903	0.917	***0.872****
	50	1.068	1.124	***1.052***	1.070	1.066	1.034	1.005	1.076	1.048	***1.001****	1.051	1.005	1.030
		0.996	1.061	***0.969***	1.004	0.997	0.958	1.058	1.055	***0.979***	1.025	0.971	0.947	***0.945****
	100	1.042	1.062	***1.017***	1.046	1.043	1.025	1.013	1.044	1.045	***1.005****	1.031	1.012	1.030
		0.998	1.033	***0.972****	1.014	1.010	0.990	1.065	1.038	***1.021***	1.027	0.987	0.987	0.991
	l	Sample: July 2005 - July 2007												
MSE	1	0.528	0.517	0.519	0.509	0.511	***0.506***	0.530	0.487	***0.485****	0.538	0.507	0.496	0.505
		0.173	0.164	0.182	***0.152***	***0.152***	0.175	0.225	***0.132****	0.136	0.215	0.159	0.157	0.156
	20	0.595	0.601	0.623	0.540	0.542	***0.535***	0.549	***0.518****	***0.518****	0.725	0.559	0.540	0.569
		0.324	0.335	0.387	0.238	0.241	***0.214***	0.257	***0.173****	0.188	0.554	0.272	0.237	0.291
	50	0.679	0.769	0.770	0.577	0.580	***0.546***	0.559	***0.523****	0.532	0.729	0.617	0.553	0.608
		0.460	0.611	0.627	0.296	0.301	***0.230***	0.263	***0.178****	0.212	0.557	0.366	0.253	0.352
	100	0.779	1.019	0.904	0.611	0.616	***0.554***	0.558	***0.516****	0.542	0.729	0.676	0.557	0.644
		0.626	1.013	0.844	0.361	0.370	***0.250***	0.270	***0.179****	0.240	0.561	0.469	0.269	0.419
MAE	1	0.543	0.527	0.554	0.512	0.514	***0.497***	0.578	***0.446****	0.465	0.561	0.525	0.506	0.517
		0.349	0.330	0.360	***0.316***	0.317	0.319	0.416	***0.255****	0.273	0.386	0.330	0.318	0.324
	20	0.657	0.652	0.691	0.574	0.578	***0.517***	0.605	***0.461****	0.516	0.805	0.606	0.586	0.632
		0.529	0.524	0.581	0.422	0.427	***0.357***	0.460	***0.290****	0.345	0.728	0.465	0.433	0.497
	50	0.749	0.822	0.832	0.629	0.634	***0.539***	0.612	***0.472****	0.548	0.805	0.682	0.599	0.678
		0.650	0.753	0.771	0.496	0.503	***0.388***	0.466	***0.303****	0.388	0.730	0.567	0.453	0.559
	100	0.842	1.007	0.937	0.680	0.687	***0.567***	0.611	***0.472****	0.577	0.806	0.752	0.609	0.722
		0.776	1.003	0.913	0.567	0.575	***0.424***	0.474	***0.310****	0.434	0.733	0.658	0.473	0.620

	l	Sample: July 2005 - April 2009												
	1	0.706	0.693	***0.648****	0.697	0.695	0.833	1.006	0.696	0.682	***0.680***	0.701	0.721	0.671
		0.441	0.474	***0.400****	0.471	0.464	0.733	1.006	0.527	***0.521***	0.562	0.479	0.580	0.465
	20	0.864	0.843	***0.835****	0.861	0.862	0.914	1.024	0.918	***0.877***	1.012	0.861	0.941	0.883
		0.797	0.767	***0.752****	0.799	0.800	0.875	1.034	***0.875***	0.819	1.017	0.793	0.909	0.824
MSE														
	50	0.962	0.981	0.968	0.966	0.965	***0.963***	1.027	0.986	***0.975***	1.013	***0.961****	0.991	0.967
		0.951	0.989	0.961	0.960	0.959	***0.948****	1.039	0.978	***0.973***	1.018	0.949	0.987	***0.957***
	100	***0.983****	0.985	0.988	0.989	0.989	0.995	1.029	1.016	***1.003***	1.013	0.986	1.012	0.992
		0.978*	0.989	0.985	0.992	0.991	0.995	1.041	1.023	***1.007***	1.018	0.984	1.016	0.991
	1	0.919	0.934	***0.896***	0.906	0.904	0.927	0.909	***0.859***	0.874	0.867	0.907	***0.855****	0.881
		0.650	0.682	***0.618***	0.643	0.639	0.743	0.895	***0.586****	0.607	0.632	0.641	0.591	0.589
	20	0.980	1.012	0.977	0.963	0.962	***0.952***	0.927	***0.901****	0.934	0.963	0.962	0.908	0.942
		0.841	0.865	0.827	0.822	***0.821***	0.833	0.934	***0.771****	0.789	0.963	0.817	0.803	0.799
MAE														
	50	1.008	1.070	1.009	0.990	***0.989***	0.944	0.932	***0.906****	0.955	0.965	0.983	0.925	0.965
		0.933	1.007	0.932	0.914	0.911	***0.856***	0.943	***0.819****	0.870	0.967	0.897	0.850	0.876
	100	1.003	1.050	1.000	0.978	0.977	***0.940***	0.939	***0.917****	0.958	0.968	0.978	0.939	0.972
		0.959	1.027	0.960	0.937	0.936	***0.893****	0.957	***0.896***	0.918	0.972	0.929	0.911	0.927

Table 9: MSE and MAE for S&P 500. For further notes see Table 5. Models - M1: FIGARCH - M2: GARCH - M3: TGARCH - M4: BMSM - M5: LMSM - M6: SV - M7: RV-ARMA - M8: RV-ARFIMA - M9: RV-LMSM - M10: RV-BNS. Combinations - C1: M2, M3, M5, M6 - C2: M7, M8, M9, M10 - C3: M2, M4, M9, M10.